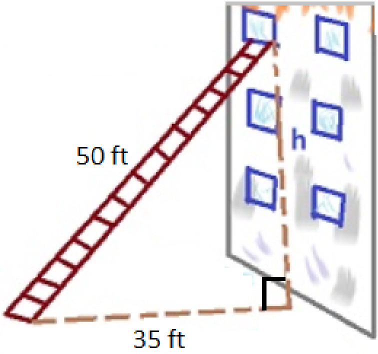
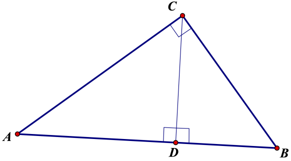
Pythagorean Theorem

If the rescue squad has a 50 ft ladder and angles it to a building, 35 ft away, how many feet high will it reach?

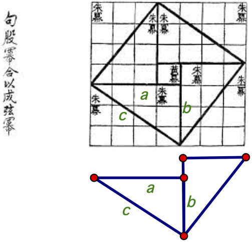


AA Similarity $\implies$ Pythagorean Theorem

Given a right $\triangle ABC$ with hypotenuse $\overline{AB}$ and shortest side $\overline{BC}$, construct a perpendicular from $\overline{AB}$ through C by I-12 and label the intersection as D. Now $\triangle ABC \sim \triangle CBD$ by AA as all right angles are congruent by Postulate 4, $\angle ACB \cong CDB$, and $\angle CBA \cong CBD$ by CN 4. By similarity of $\triangle ABC$ and $\triangle CBD$, since the ratio of the hypotenuse to the shorter base side must be the same, $\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{BC}}{\overline{BD}}$. Cross multiply to see $\overline{AB}\,\overline{BD} = \overline{BC}^2$. Also, $\triangle ABC \sim \triangle ACD$ by AA as $\angle ACB \cong CDA$ by Postulate 4 and $\angle CAB \cong CAD$ by CN 4. $\triangle ABC \sim \triangle ACD$ so the ratio of the hypotenuse to the longer base side must be the same, $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AC}}{\overline{AD}}$. Cross multiply: $\overline{AB}\,\overline{AD} = \overline{AC}^2$. By CN 2, $\overline{AB}\,\overline{BD} + \overline{AB}\,\overline{AD} = \overline{BC}^2 + \overline{AC}^2$. Factor $\overline{AB}$ so that $\overline{AB}(\overline{BD} + \overline{AD}) = \overline{AB}^2$ by CN 4. So $\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$ by CN 1.



Why is the Pythagorean Theorem True?



周髀算經 or *Zhoubi Suanjing*

Use the labels and the puzzle...

large square has side c

small square has side

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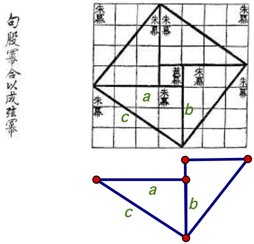

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Use the labels and the puzzle...

large square has side c

small square has side $a - b$

area of the large square = area small square + 4 area of triangle

$c^2 = (a-b)^2 + 4(\frac{ab}{2})$

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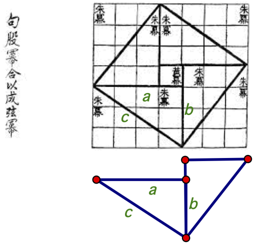

周髀算經 or *Zhoubi Suanjing*

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$$c^2 = a^2 - 2ab + b^2 + 2ab$$

$$c^2 = a^2 + b^2$$

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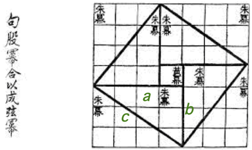

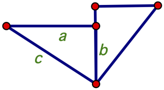

周髀算經 or *Zhoubi Suanjing*

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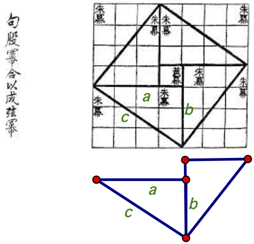
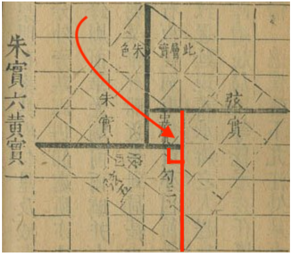
$c^2 = a^2 + b^2$

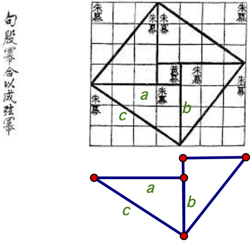
Proof: Where should we start? Why do we have squares?

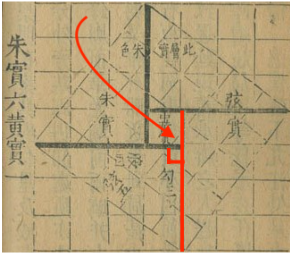




周髀算經 or *Zhoubi Suanjing*

Given a non-isosceles ◸ with hypotenuse *c* opposite a right angle, long base side *a* and short side *b*, to show it satisfies the Pythagorean theorem, first translate and rotate to form 3 congruent copies as in the picture. To show the inside space is a square, notice each 90° angle in a triangle on side *b* is also on a side *a* of an adjacent triangle that is broken up by a side of the small square so by I-13 it has 90° angles





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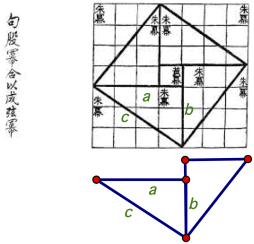
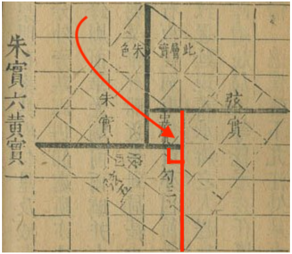
Given a non-isosceles ◸ with hypotenuse c opposite a right angle, long base side a and short side b, to show it satisfies the Pythagorean theorem, first translate and rotate to form 3 congruent copies as in the picture. To show the inside space is a square, notice each 90° angle in a triangle on side b is also on a side a of an adjacent triangle that is broken up by a side of the small square so by I-13 it has 90° angles and its sides are $a - b$, so it is a square by Def 22. To show the large figure is a square, it has sides c and each angle is made up of the sum of angles opposite a and b in adjacent triangles.





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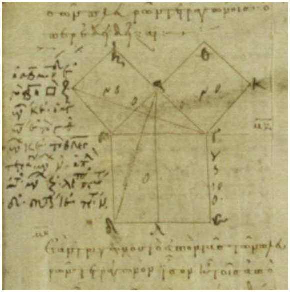
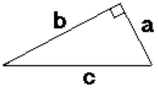
Why is the Pythagorean Theorem True?

https://www.youtube.com/watch?v=CAkMUdeB06o

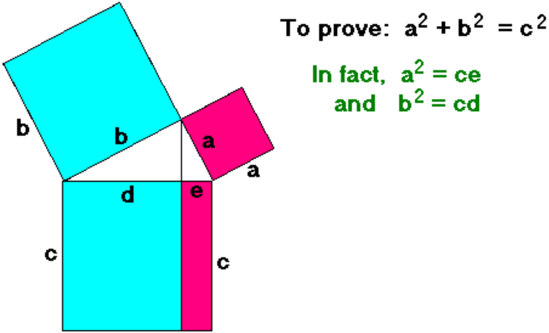


Translation of *Euclid's Elements*

To prove: $a^2 + b^2 = c^2$



by Dave Lantz at Colgate



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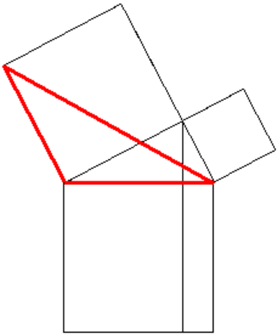

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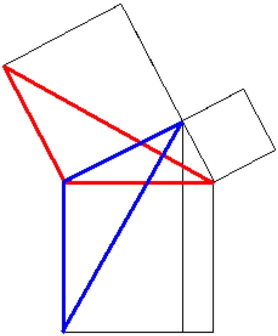

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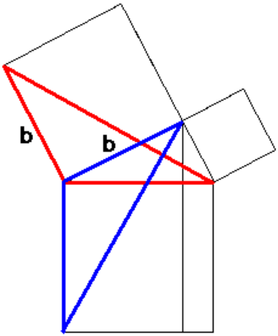

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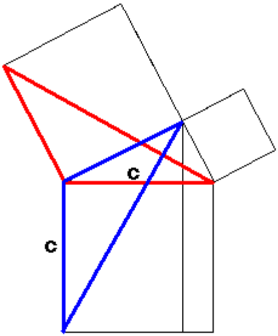

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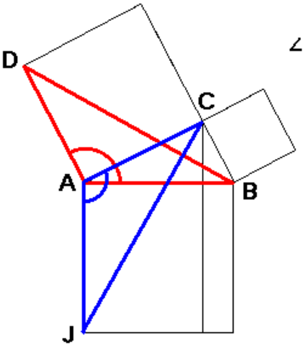
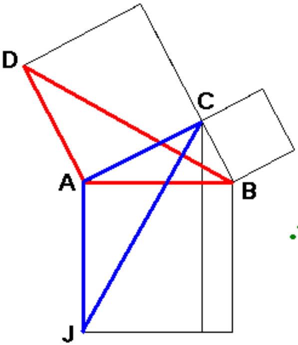


$\angle DAB$

$= \angle CAB$ + a right angle

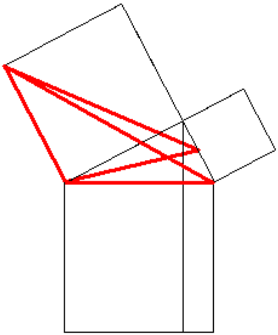
$= \angle CAJ$

by Dave Lantz at Colgate



$\therefore \triangle DAB \cong \triangle CAJ$

by Dave Lantz at Colgate



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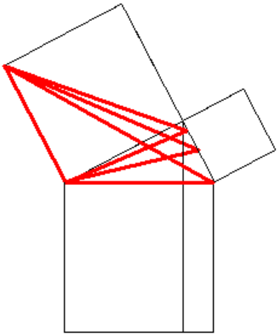

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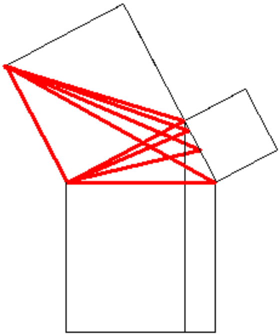

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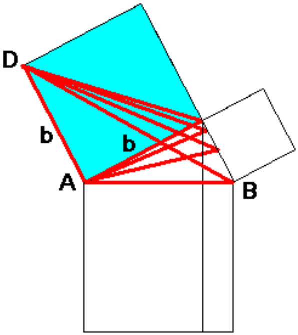
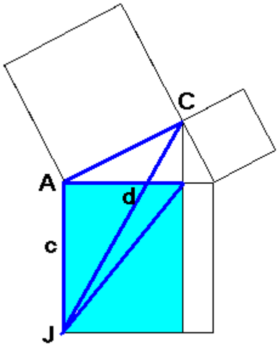


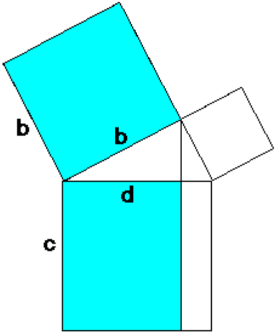
$\text{area}(\triangle DAB) = \frac{1}{2} b^2$

by Dave Lantz at Colgate



Similarly,

area($\triangle CAJ$) = $\frac{1}{2}cd$



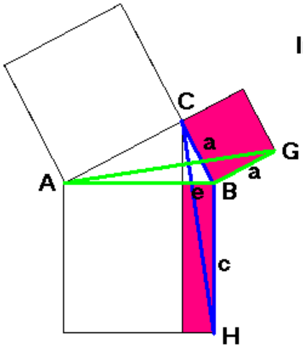
$\text{area}(\triangle DAB) = \frac{1}{2} b^2$

Similarly,

$\text{area}(\triangle CAJ) = \frac{1}{2} cd$

$\therefore\ b^2 = cd$

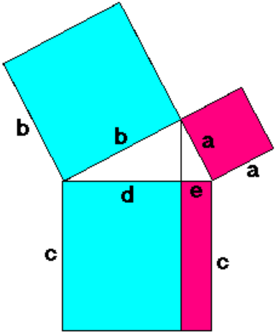
by Dave Lantz at Colgate



In the same way,

$$a^2 = 2\ \text{area}(\triangle GAB)$$
$$= 2\ \text{area}(\triangle CHB)$$
$$= ce$$

by Dave Lantz at Colgate

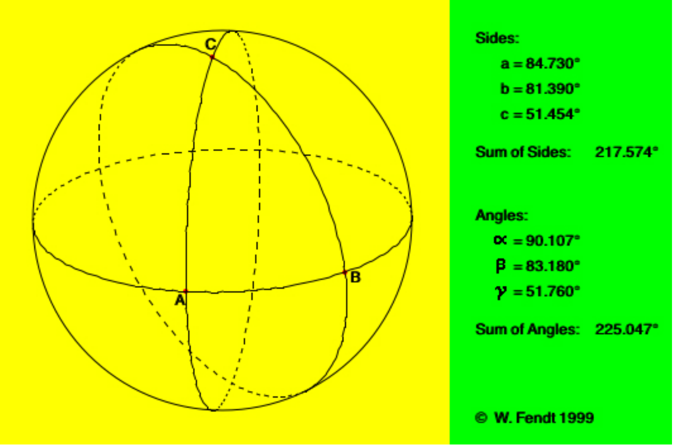


Therefore:

$$b^2 + a^2 = cd + ce$$
$$= c(d+e)$$
$$= c^2$$

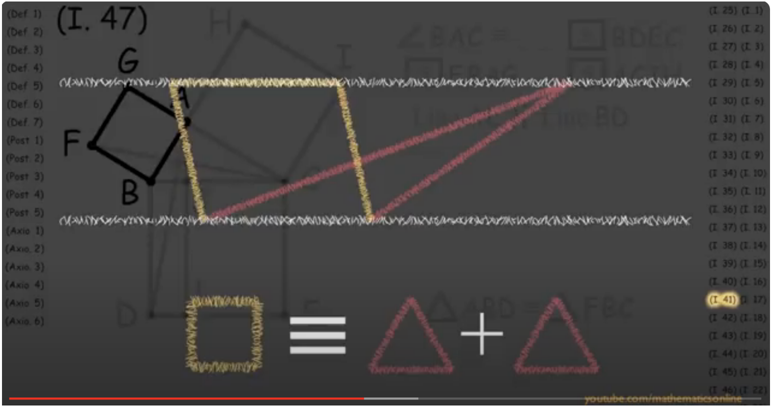
QED!

by Dave Lantz at Colgate

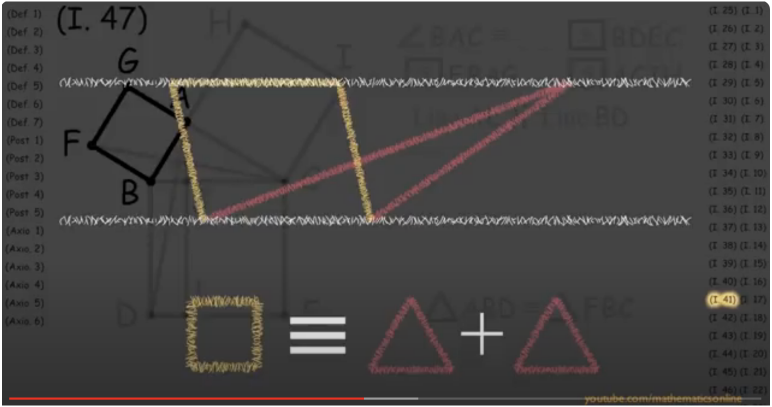


$AB^2 + AC^2 \approx 51.454^2 + 81.390^2 > 84.730^2 = BC^2$

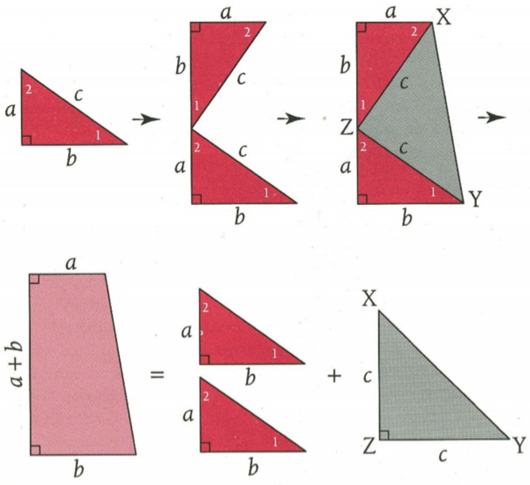
$c_{\text{Euclidean}}$ is too long for sphere: $a^2 + b^2 = c^2_{\text{Euclidean}} > c^2_{\text{sphere}}$



- What are aspects you like about this video from mathmaticsonline?



- What are aspects you like about this video from mathmaticsonline?
- axiom 2 versus
  CN 2 (equals are added to equals $\implies$ wholes are equal)
- parallelogram and square have equal areas
  they are not congruent as figures

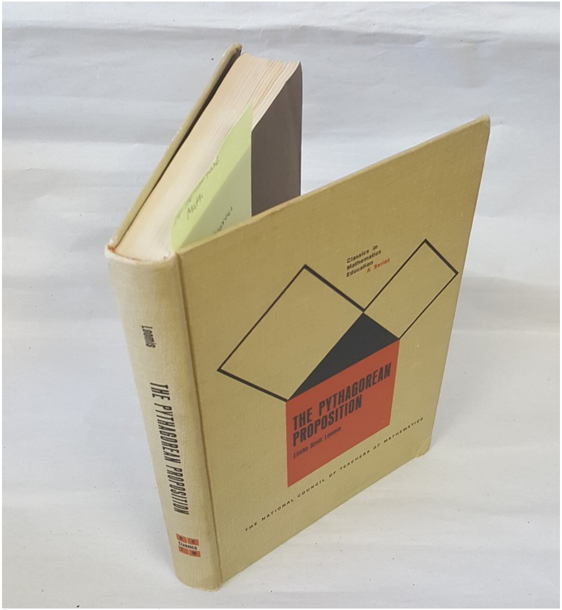


$$\tfrac{1}{2}(a+b)(a+b) = 2(\tfrac{1}{2}ab) + \tfrac{1}{2}c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$\textit{Therefore, } a^2 + b^2 = c^2$$

https://www.maa.org/press/periodicals/convergence/mathematical-treasure-james-a-garfields-proof-of-the-pythagorean-theorem



Elisha Scott Loomis
367 proofs of the Pythagorean theorem

Where Else is Pythagorean Theorem Historically?

Babylonian cuneiform Plimpton 322

cuneiform

Baudhayana sutras (Vedic Sanskrit texts)

दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यग् मानी च यत् पृथग् भूते कुरूतस्तदुभयं करोति ॥

A rope stretched along the length of the diagonal
produces an area
which the vertical and horizontal sides make together.

Egypt 3, 4, 5 right triangles

Extensions of the Pythagorean Theorem

- non-Euclidean geometry
- squares to other regular polygons on the right triangle
- squares to other algebraic powers