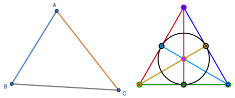
Axiomatic Systems, Measurement, and Constructions

Models of incidence geometry:



- Axiom 1) For each two distinct points there exists a unique line on both of them.
- Axiom 2) For every line there exists at least two distinct points on it.
- Axiom 3) There exist at least three distinct points.
- Axiom 4) Not all points lie on the same line.

Proposition: l and m are two distinct lines that meet $\implies$ they meet at a unique point

To prove $A \Rightarrow B$ we assume A and prove B

To prove $A \not\Rightarrow B$: $A \cap \sim B$

Incidence Geometry Proof by Contradiction

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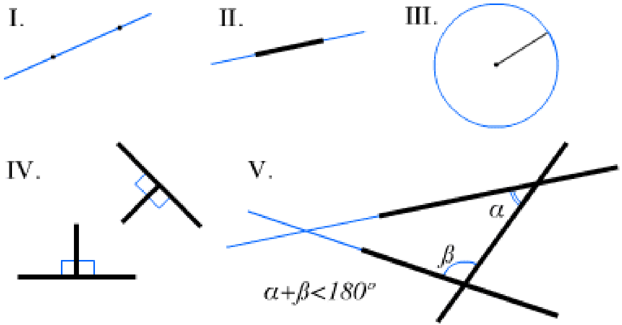
Proposition: l and m are two distinct lines that meet $\implies$ they meet at a unique point

Proof: Assume that l and m are two distinct lines that meet. For contradiction, assume that they meet in 2 distinct points, P and Q. However, now P and Q have the 2 distinct lines l and m on both of them, contradicting Axiom 1. Thus l and m meet at a unique point, as desired.

Euclid's 5 Postulates

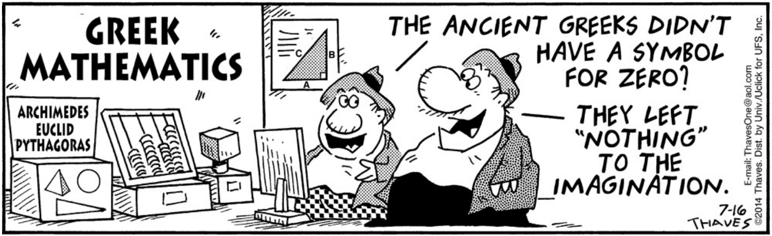
In Euclid's *Elements*, the axioms are called postulates.



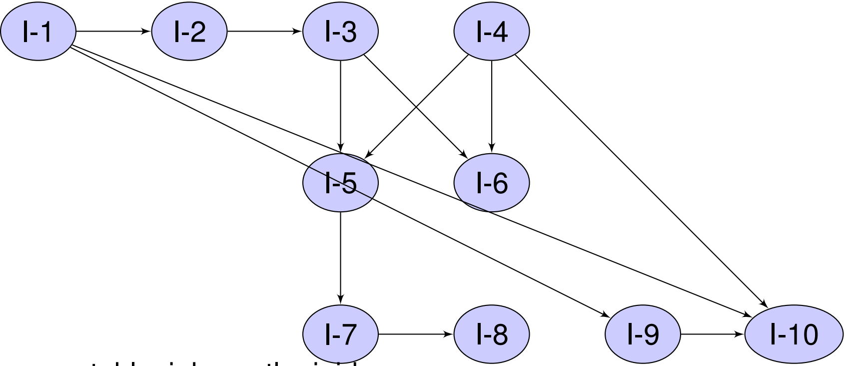
https://www.storyofmathematics.com/hellenistic_euclid.html

Euclid's Elements I-1

I-1: On a given finite straight line, to construct an equilateral triangle (with only straightedge, compass, and intersection)



Propositions, Assumptions and Applications



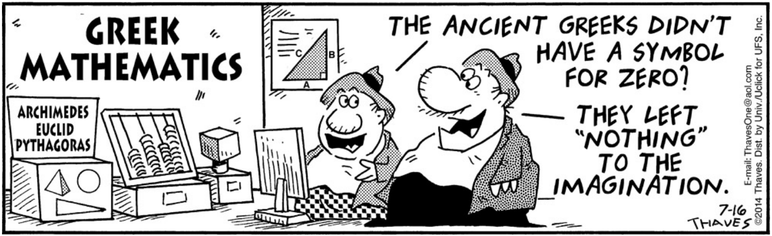
stable, inherently rigid

Six Huzita or Justin Folding Axioms for Origami

- Given points $P \neq Q$, there is a unique fold that passes through both of them.
- Given $P \neq Q$, there is a unique fold that places P onto Q.
- Given lines l and m, there is a fold that places l onto m.
- Given P and l, there is a unique fold perpendicular to l that passes through point P.
- Given $P \neq Q$ and l, there is a fold that places P onto l and passes through Q.
- Given $P \neq Q$ and $l \neq m$, there is a fold that places P onto l and Q onto m.

Euclid's Elements I-1 in IGS

I-1: On a given finite straight line, to construct an equilateral triangle (with only straightedge, compass, and intersection)



Let $\overline{AB}$ be a line segment. We'll construct an equilateral triangle with $\overline{AB}$ as the base...

Proof of *Euclid's Elements* I-1

Let $\overline{AB}$ be a line segment. We'll construct an equilateral triangle with $\overline{AB}$ as the base. Use postulate

Proof of *Euclid's Elements* I-1

Let $\overline{AB}$ be a line segment. We'll construct an equilateral triangle with $\overline{AB}$ as the base. Use postulate 3 to draw a circle with center A and radius AB. Similarly construct a circle with center B and radius BA.

Proof of *Euclid's Elements* I-1

Let $\overline{AB}$ be a line segment. We'll construct an equilateral triangle with $\overline{AB}$ as the base. Use postulate 3 to draw a circle with center A and radius AB. Similarly construct a circle with center B and radius BA. Construct an intersection of the two circles, C. *

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Proof of *Euclid's Elements* I-1

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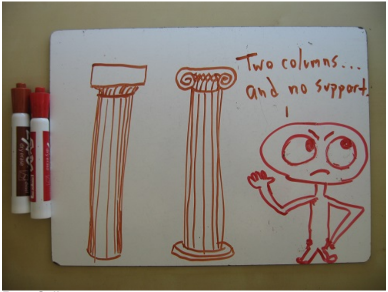
Why Paragraph Proofs?

Ben Orlin: A good proof contains not only bare statements of fact, but connective tissue of explanation. In a two-column proof, the organic matter that holds the argument together is flushed away, and replaced with a right-hand column full of terse bullet points that students may use without understanding



Ben Orlin, https://mathwithbaddrawings.com

Professional association against two-column proof

Euclid's Elements I-11

I-11: To draw a straight line at right angles to a given straight line from a given point on it (with only straightedge, compass, and intersection)

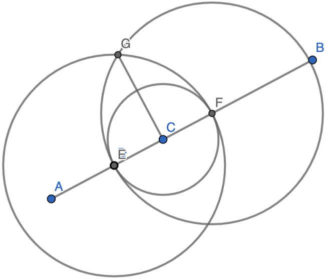
https://i.redd.it/xu89vcweusc11.jpg

Purr-pendicular

I-13, I-46, and I-48
engineering

Euclid's Elements I-11

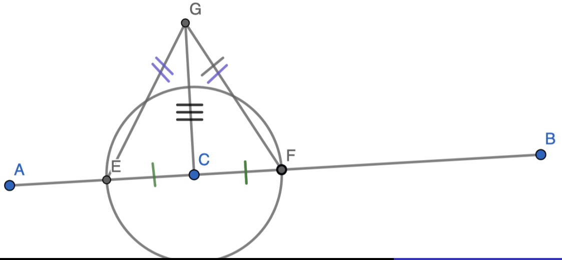
Part 1 of Proof: Let $\overline{AB}$ be a line segment with C on it. We'll prove we can construct a line through C perpendicular to $\overline{AB}$. Choose a point on the shorter of AC or BC (or on one of them if they are the same length) by CN 5 & Def 4. Construct circle center C and radius CE by postulate 3. Let E and F be the intersection of the circle with $\overline{AB}$. Construct G as a vertex of the equilateral triangle by I-1. Use postulate 1 to construct $\overline{GC}$.



Euclid's Elements I-11

Part 2 of Proof: To show that $\overline{GC}$ is perpendicular to $\overline{AB}$, compare triangles ECG and FCG, after completing the segments $\overline{EG}$ and $\overline{FG}$ by postulate 1. Notice $CE = CF$ by Def 15 as they are radii of the circle centered at C. In addition $CG = CG$ by CN4. Also, $EG = EF = FG$ because they are constructed using I-1, the equilateral triangle. So $\triangle ECG \cong \triangle FCG$ by SSS congruence in I-8. Thus, $\angle GCE \cong \angle GCF$. By definition 10 the angles must be right.

https://www.geogebra.org/geometry/cr9r72vg



Federico Ardila's Axioms

- Mathematical potential is distributed equally among different groups, irrespective of geographic, demographic, and economic boundaries.
- Everyone can have joyful, meaningful, and empowering mathematical experiences.
- Mathematics is a powerful, malleable tool that can be shaped and used differently by various communities to serve their needs.
- Students deserve to be treated with dignity and respect.

Ardila-Mantilla, Federico (2016). Todos Cuentan: Cultivating Diversity in Combinatorics. *Notices of the American Mathematical Society* 63(10) 1164–1170.

Wile E. Coyote is trapped in the following axiom system:

- A1: Coyotes and roadrunners live on the surface of a perfectly round planet.
- A2: Coyotes only begin chasing roadrunners exactly 2 seconds after the roadrunner passes them.
- A3: Coyotes can only catch roadrunners if they can catch up to them after having chased them.
- A4: Roadrunners run faster than coyotes.
- A5: Coyotes stop chasing roadrunners when they disappear from view.
- A6: All coyotes have 20/20 vision.

If needed, can you add other axioms to the system, which are consistent with it, to ensure I will always catch the roadrunner?
Help me - you're my only hope!
Wile E. Coyote