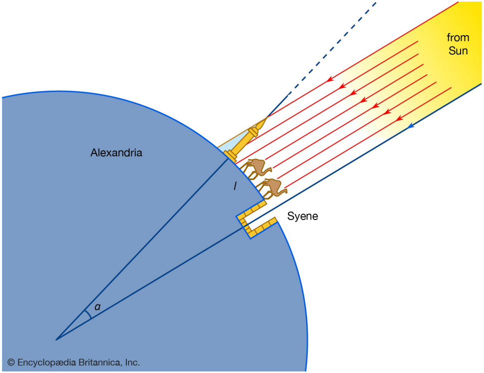
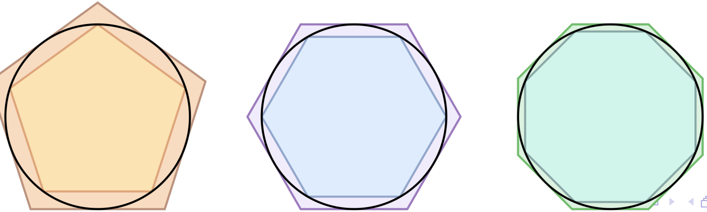
Measurements Pre Analytic/Metric Geometry?

- Eratosthenes and circumference of earth
- Archimedes polygonal method for circumference and π
- Archimedes' area of circle by exhaustion
- Archimedes' surface area of a sphere
- Archimedes' volume of a sphere (Cavalieri's principle)
- area of spherical $\triangle$ and sum of the angles

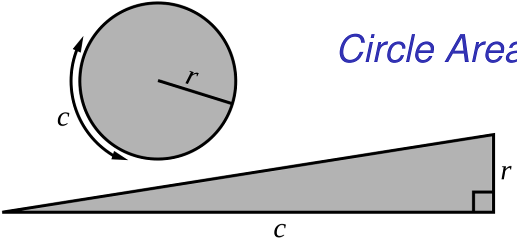


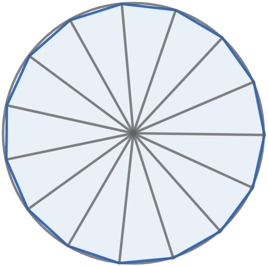
"Pi what squared? Long John, you should be able to get this."

https://www.newyorker.com/online/blogs/cartoonists/pi-what-squared.jpg

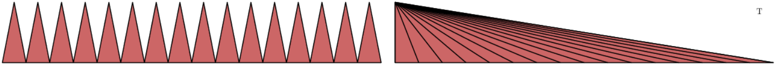


Circle Area





If $C > T$, where area circle $= C$ and area triangle $= T$ then inscribe a polygon inside the circle with area P so that $T < P < C$. Dissect the polygon into triangles. height $<$ radius and sum of the widths (perimeter) $<$ circumference because it is inside. Hence $P < T$, a contradiction to $T < P$.



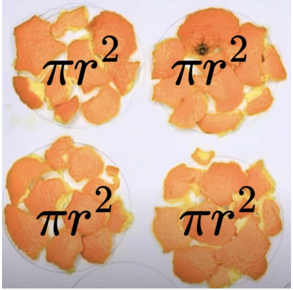
http://www.ams.org/publicoutreach/feature-column/fc-2012-02

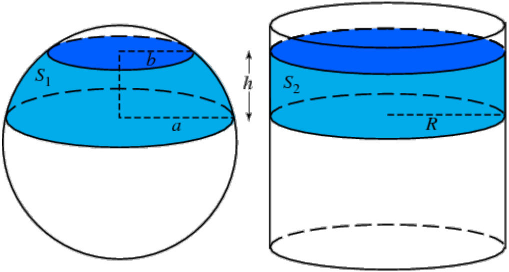
Surface Area of Sphere

4 times the area of a great circle of the sphere $= 4\pi r^2$

Surface Area of Sphere

4 times the area of a great circle of the sphere $= 4\pi r^2$





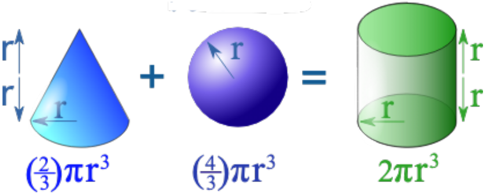
Mathberry Lane http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

If a polygon is inscribed in a circle and revolved about an axis, forming a solid of revolution inside a sphere, then the surface area $<$ surface area of sphere.

Volume of a Sphere with Sand

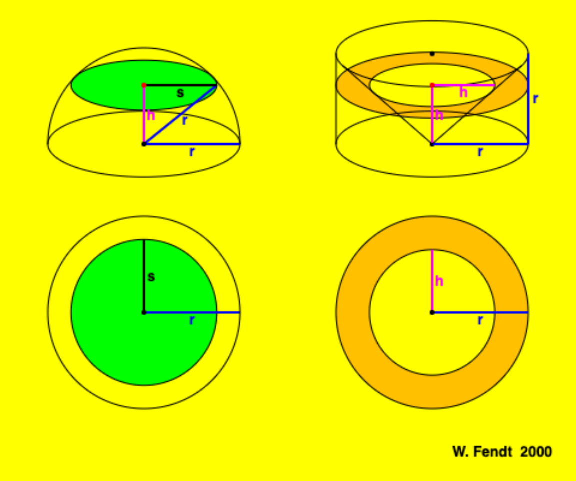
- Fill up the sphere with sand and pour it into the cylinder. Approximately what fraction of the cylinder does the sphere take up?
- How many cones of sand fill up the cylinder? What fraction of the cylinder does the cone take up?

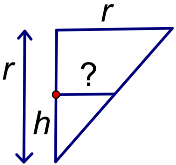
cone + sphere volumes: $\frac{1}{3}\pi r^2(h) + \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2(2r) + \frac{4}{3}\pi r^3$
cylinder volume: $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$



https://www.mathsisfun.com/geometry/cone-sphere-cylinder.html

Volume of a Sphere: Cavalieri's Principle



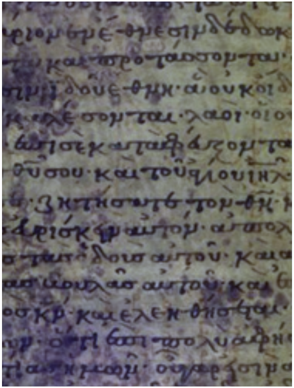


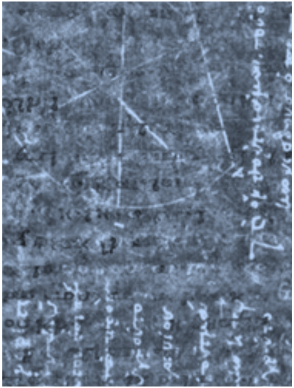
https://www.walter-fendt.de/html5/men/volumesphere_en.htm

$\pi r^2 - \pi h^2 = \pi(r^2 - h^2) = \pi\sqrt{(r^2 - h^2)}^2$

What familiar theorems are assumed?

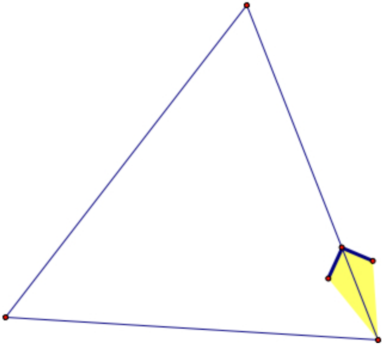
Archimedes: Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards.
[*The Method* in *The Works of Archimedes* translated by Heath]
http://www.archimedespalimpsest.org/





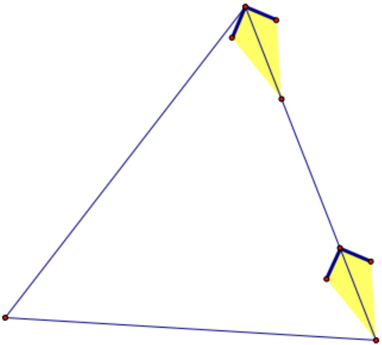
Angle Sum Measurements in Various Geometries

- Lay out a triangle with masking tape
- Pick a vertex to begin your triangle walk.
  Note the vertex and which way you are facing.



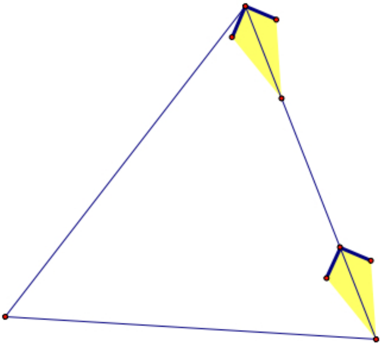
Walking a Euclidean Angle Sum

- Start walking along your triangle, keeping the center of your body on the boundary of the triangle.



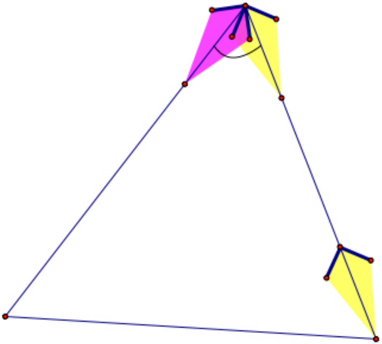
Walking a Euclidean Angle Sum

- When you get to a turn (one of the angles of the triangle), turn your body so that it sweeps the interior angle of the triangle (careful!). You may be walking backwards for a time.



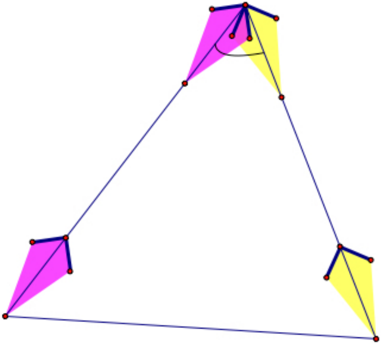
Walking a Euclidean Angle Sum

- When you get to a turn (one of the angles of the triangle), turn your body so that it sweeps the interior angle of the triangle (careful!). You may be walking backwards for a time.



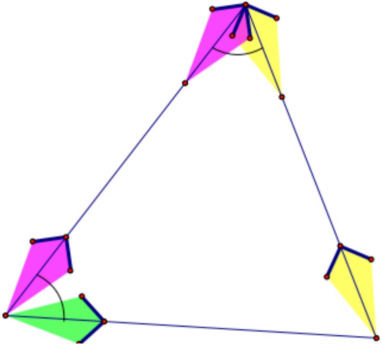
Walking a Euclidean Angle Sum

- When you get to a turn (one of the angles of the triangle), turn your body so that it sweeps the interior angle of the triangle (careful!). You may be walking backwards for a time.



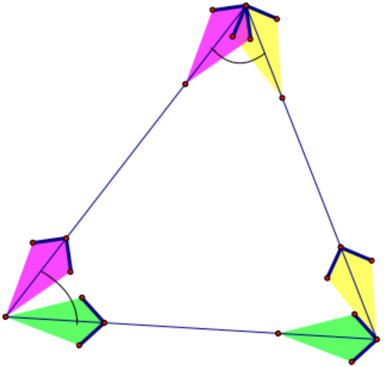
Walking a Euclidean Angle Sum

- When you get to a turn (one of the angles of the triangle), turn your body so that it sweeps the interior angle of the triangle (careful!). You may be walking backwards for a time.



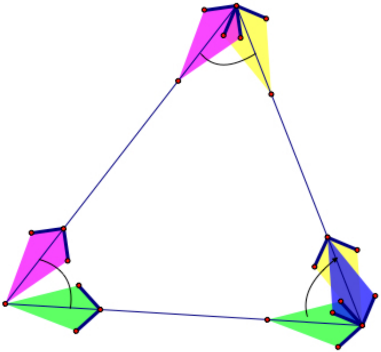
Walking a Euclidean Angle Sum

- When you get to a turn (one of the angles of the triangle), turn your body so that it sweeps the interior angle of the triangle (careful!). You may be walking backwards for a time.



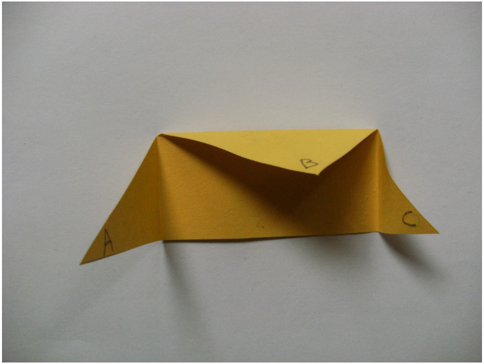
Walking a Euclidean Angle Sum

- Sweep out the last interior angle to finish your angle sum walk.
- The change in direction in your body from start to finish is the sum of the angles in this triangle.



Folding an Angle Sum Extrinsically

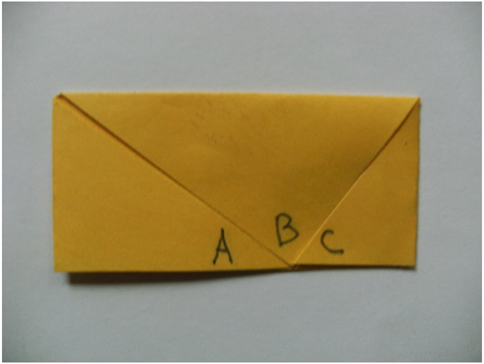
- Rip a triangle from paper.
- Fold one angle to bring it down to the base by using a fold parallel to the base.
- Fold the other angles in



http://mathonthemckenzie.blogspot.com/2013/12/180.html

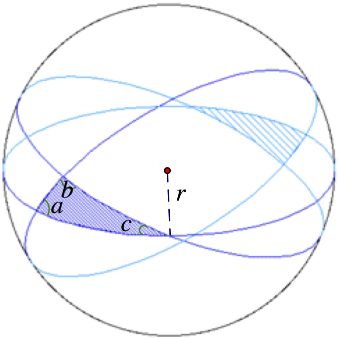
Folding an Angle Sum Extrinsically

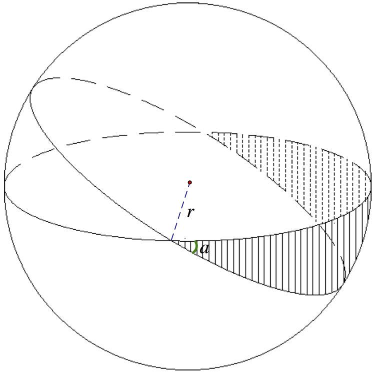
- Notice the angles fit to take up the entire space along the base and this gives us the angle sum.



http://mathonthemckenzie.blogspot.com/2013/12/180.html

Sum of the Angles in a Triangle on a Sphere



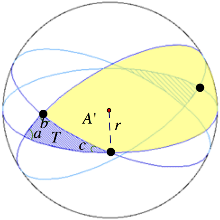


area of lune of angle a radians
$= \frac{a}{2\pi} \times$ surface area of sphere
$= \frac{a}{2\pi} 4\pi r^2 = 2ar^2$

Sum of the Angles in a Triangle on a Sphere

area of lune of angle a radians= $\frac{a}{2\pi} \times$ surface area of sphere

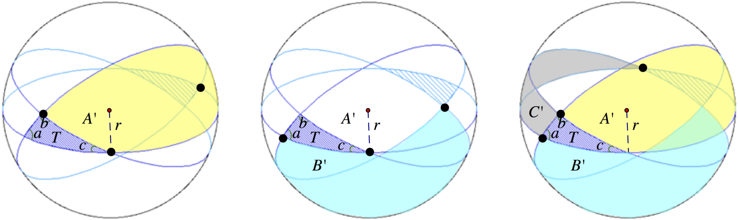
$$= \frac{a}{2\pi} 4\pi r^2 = 2ar^2$$



Sum of the Angles in a Triangle on a Sphere

area of lune of angle a radians= $\frac{a}{2\pi} \times$ surface area of sphere

$$= \frac{a}{2\pi} 4\pi r^2 = 2ar^2$$

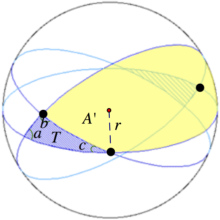


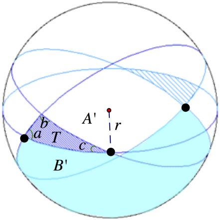
$T + A' = 2ar^2$ $\quad$ $T + B' = 2br^2$ $\quad$ $T + C' = 2cr^2$

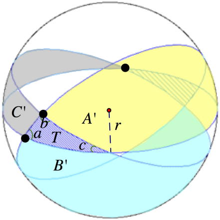
Sum of the Angles in a Triangle on a Sphere

area of lune of angle a radians $= \frac{a}{2\pi} \times$ surface area of sphere

$$= \frac{a}{2\pi} 4\pi r^2 = 2ar^2$$





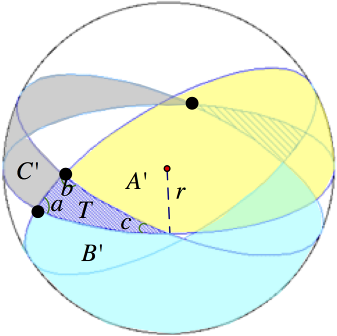


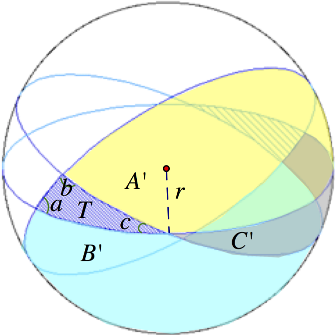
$T + A' = 2ar^2$ $\qquad$ $T + B' = 2br^2$ $\qquad$ $T + C' = 2cr^2$

$$3T + A' + B' + C' = 2ar^2 + 2br^2 + 2cr^2$$

Sum of the Angles in a Triangle on a Sphere

equation 1: $3T + A' + B' + C' = 2ar^2 + 2br^2 + 2cr^2$



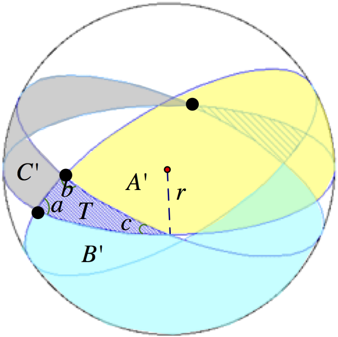


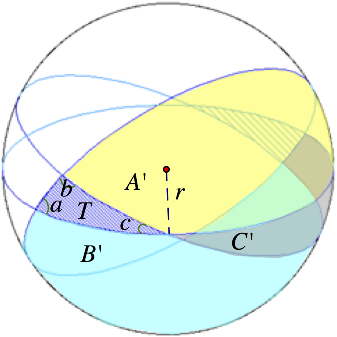
equation 2:

$T + A' + B' + C' = \text{hemisphere} = \frac{4\pi r^2}{2} = 2\pi r^2$

Sum of the Angles in a Triangle on a Sphere

equation 1: $3T + A' + B' + C' = 2ar^2 + 2br^2 + 2cr^2$





equation 2:

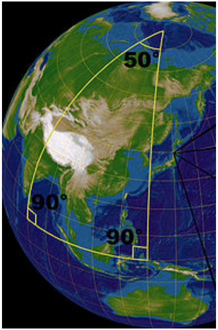
$T + A' + B' + C' = \text{hemisphere} = \frac{4\pi r^2}{2} = 2\pi r^2$

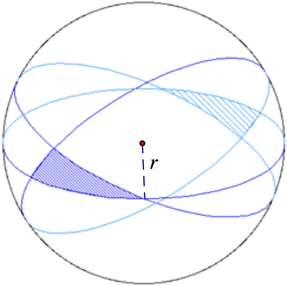
equation 1 $-$ equation 2:

$2T = 2r^2(a + b + c - \pi)$

area of the triangle $= r^2(\text{sum of the angles} - \pi)$

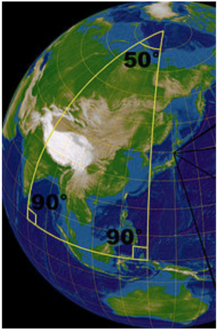
Detecting Angle Sum in Earth Triangle

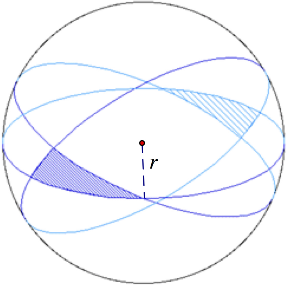




$$\text{sum of the angles} - \pi = \frac{\text{area of the triangle}}{r^2}$$

Detecting Angle Sum in Earth Triangle



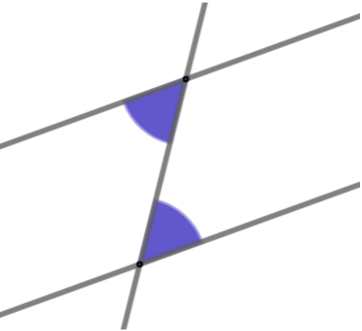


$$\text{sum of the angles} - \pi = \frac{\text{area of the triangle}}{r^2}$$

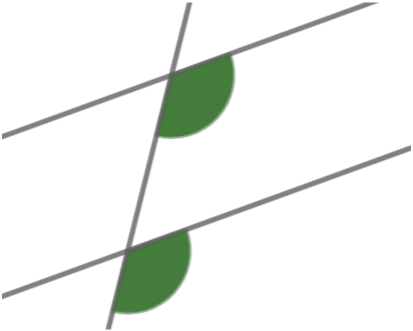
$$\frac{1}{3959^2} \approx 6.38 \times 10^{-8}$$

$$\frac{82277}{3959^2} \approx 0.005$$

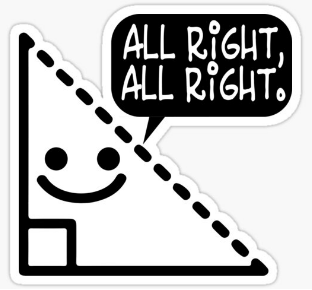
$$\frac{196{,}000{,}000/8}{3959^2} \approx 1.57$$



alternate interior angles



corresponding angles



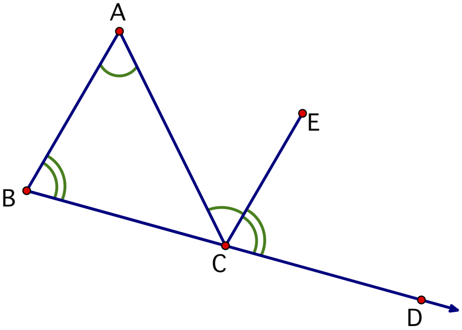
https://www.redbubble.com/

1st Part of Euclidean Proof of I-32

Let ABC be a triangle. To show the angle sum is 2 right angles, extend BC by Postulate 2 and let D be a point on it so C is between B and D. Next construct $\overline{CE}$ parallel to $\overline{AB}$ through C by

1st Part of Euclidean Proof of I-32

Let ABC be a triangle. To show the angle sum is 2 right angles, extend BC by Postulate 2 and let D be a point on it so C is between B and D. Next construct $\overline{CE}$ parallel to $\overline{AB}$ through C by I-31. Notice that alternate interior angles $\angle BAC \cong \angle ACE$ by I-29 since the transversal $\overline{AC}$ cuts the parallels $\overline{AB}$ & $\overline{CE}$. Similarly by I-29, $\overline{BD}$ cuts $\overline{AB}$ & $\overline{CE}$, so the corresponding angles $\angle ABC \cong \angle ECD$.

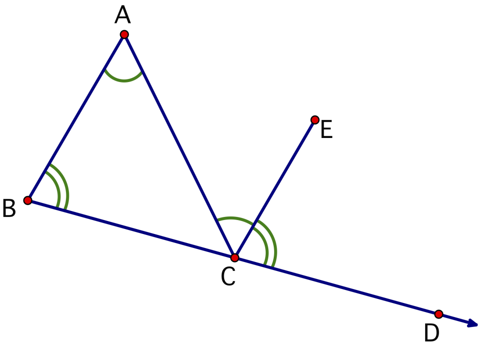


Euclidean Proof of I-32 continued

Apply CN2 to see that $\angle BAC + \angle ABC \cong \angle ACE + \angle ECD$.
Also, by CN4, $\angle ACD \cong \angle ACE + \angle ECD$. So, by CN1,
$\angle ACD \cong \angle BAC + \angle ABC$.

Euclidean Proof of I-32 continued

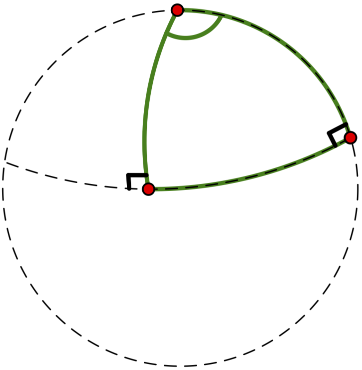
Apply CN2 to see that $\angle BAC + \angle ABC \cong \angle ACE + \angle ECD$. Also, by CN4, $\angle ACD \cong \angle ACE + \angle ECD$. So, by CN1, $\angle ACD \cong \angle BAC + \angle ABC$. Thus $\angle ACB + \angle ACD \cong \angle ACB + \angle BAC + \angle ABC$ by CN2. In addition, we know that $\angle ACB + \angle ACD$ is 2 right angles by I-13, so the interior angles of triangle ABC are also 2 right angles by CN1.



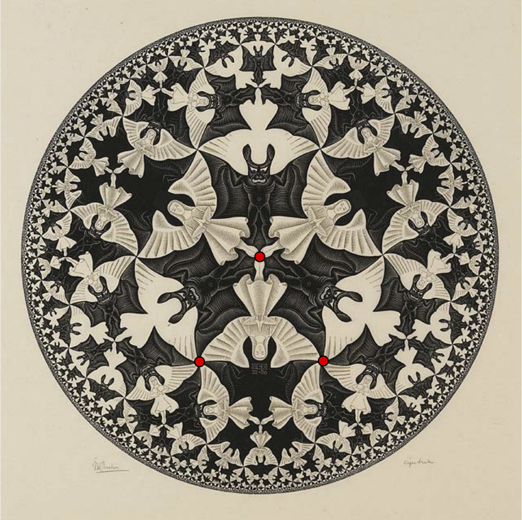
I-32 on the Sphere

- What first goes wrong with the Euclidean proof of I-32 on the sphere?

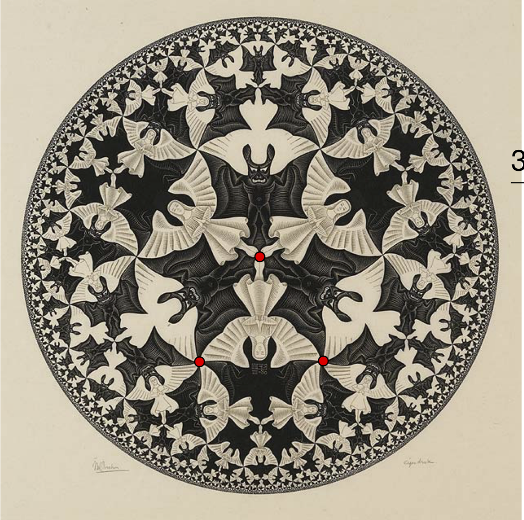
$$\text{sum of the angles} - \pi = \frac{\text{area of the triangle}}{r^2}$$



I-32 in Hyperbolic Geometry



I-32 in Hyperbolic Geometry



$$\frac{360}{8} + \frac{360}{8} + \frac{360}{6}$$

Circle Limit 4: Heaven and Hell by M.C. Escher, 1960