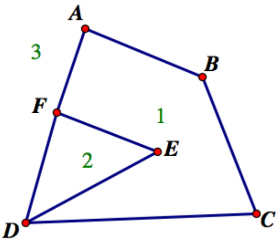
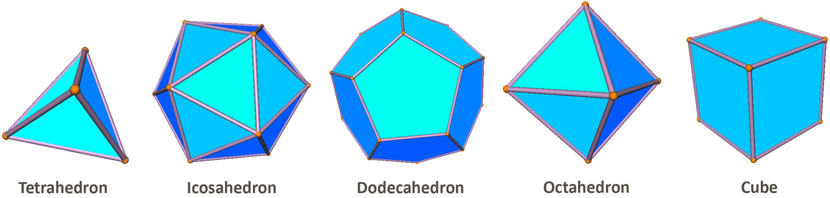
V-E+F Experiment

- Draw a few dots (vertices)
- Connect the dots with lines, subject to the following rules:
  - lines may not cross each other as they move from dot to dot
  - every dot must be connected to every other dot through a sequence of lines
  - every region must topologically be a disk with no holes
- Compute
  Vertices (V) - Edges (E) + Faces Separated by Edges (F)
  [Do not forget to count the outside as a region for F too.]



Polyhedra V-E+F and Symmetries

- What is Vertices (V) - Edges (E) + Faces (F) for the regular polyhedra?
- Where is the symmetry of a $\frac{2\pi}{3}$ rotation for each polyhedra? Describe the axis of rotation in each case.



http://www.princeton.edu/pr/pictures/l-r/packingproblem/pu-platonic-solids.jpg

Regular Polyhedra Symmetries

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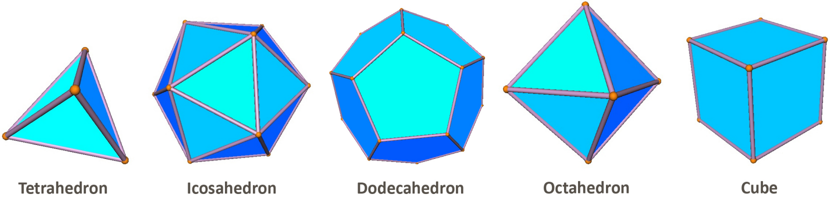

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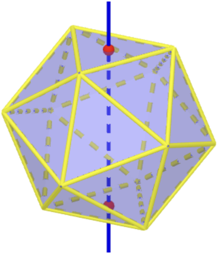
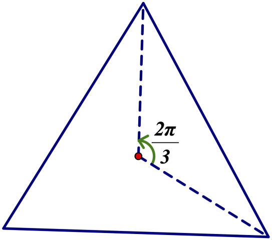




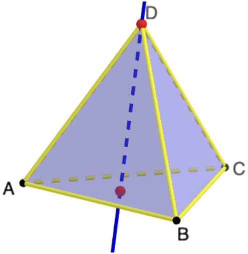
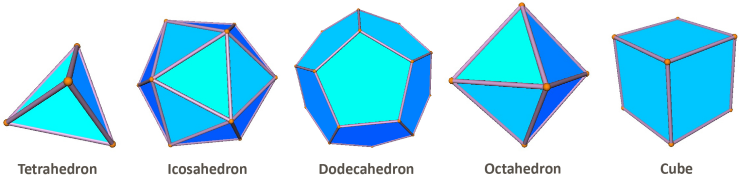


Image 1 and 3 from https://www.geogebra.org/m/PA7zzxHa, https://www.geogebra.org/m/PgyzAXRP

Platonic Solids

There are only five regular polyhedra, but why?

So their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey. [Plato, The Timaeus]



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Exactly 5 Regular Euclidean Polyhedra Part 1

Given a regular polyhedra with n total polygonal faces and p k-sided faces touching at a vertex, we'll show it must be a Platonic solid. Let E be the total edges and V the total vertices.

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multiply by $\frac{2}{k}$

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multiply by $\frac{2}{k}$ $\qquad \frac{2}{p} + \frac{2}{k} > 1$

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p k-sided faces touch at a vertex

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Euclidean polyhedra have $p \geq 3$

and regular planar polygons have $k \geq 3$

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$p = 3$ and $k \geq 6$

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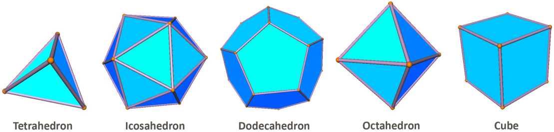
$p = 5$ and $k = 3$ icosahedron

$p \geq 6$ and $k = 3$ doesn't satisfy inequality

These are the only possibilities.



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Spherical Icosahedron

- William O. Gustafson / Uwe Meffert
- 23 563 902 142 421 896 679 424 000 combinations
- $V = 12, E = 30, F = 20$

Infinitely Many Regular Spherical Polyhedra

$$\frac{2}{p} + \frac{2}{k} > 1$$

p k-sided faces touch at a vertex

Euclidean polyhedra $p \geq 3$ and regular planar polygon $k \geq 3$

k=3 and p=3 tetrahedron

k=3 and p=4 octahedron

k=3 and p=5 icosahedron

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Could k be 2 on a sphere?

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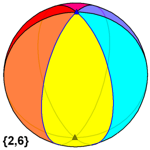


Hexagonal hosohedron

Infinitely Many Regular Spherical Polyhedra

$$\frac{2}{p} + \frac{2}{k} > 1$$

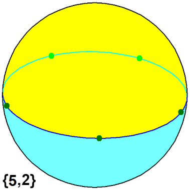
p *k*-sided faces touch at a vertex
Could *p* be 2 on a sphere?

Infinitely Many Regular Spherical Polyhedra

$$\frac{2}{p} + \frac{2}{k} > 1$$

p k-sided faces touch at a vertex

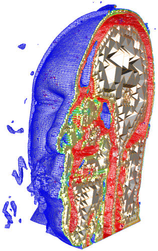
Could p be 2 on a sphere?

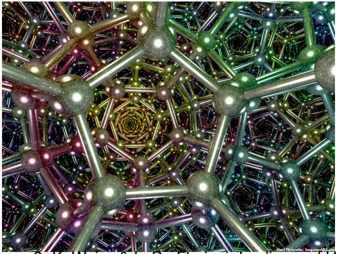


Pentagonal dihedron

Measuring Curvature at a Vertex

Structure of Viruses Approximations Shape of Universe





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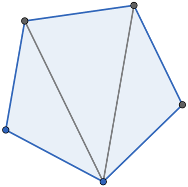
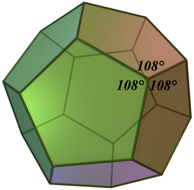
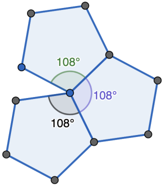
Angle defect at a vertex = 360° − sum angles at a vertex







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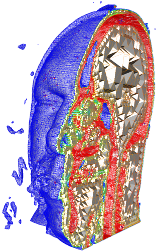

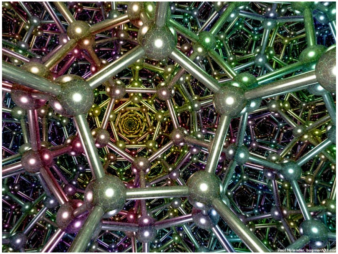

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Polyhedron	Angle Defect	V	Total Angle Defect
Dodecahedron	36°	20	$20 \times 36^\circ = 720^\circ$
flat soccer ball (truncated icosahedron)	12°	60	$12 \times 60^\circ = 720^\circ$

Why is the Total Angle Defect 720°?

angle defect at a vertex = $2\pi -$ sum angles at a vertex

total angle defect =
$\sum_V$ angle defect at a vertex=

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sum of angles in 1 face =

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sum of all the angles=

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Recall that all the sides double counts along the edges E so
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total angle defect = $2\pi V - (2\pi E - 2\pi F)$ = $2\pi(V - E + F)$
geometric combinatorics