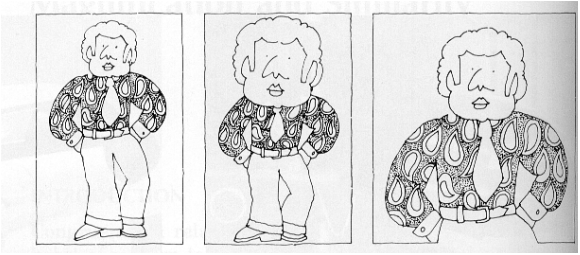
Congruence and Similarity: Same Shape?



congruence

corresponding sides congruent

corresponding angles congruent

similarity

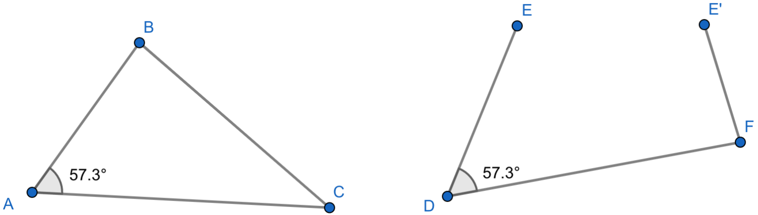
corresponding sides proportional:

same proportionality constant

corresponding angles congruent

Congruence and Similarity: Euclidean SAS

- SAS: Two sides of the broken triangle DEF are the same length as the corresponding sides in $\triangle ABC$ ($\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$). The included angle is congruent ($\angle BAC \cong \angle EDF$). How many triangles can we create? Must they be congruent? If not, must they be similar? Why or why not?

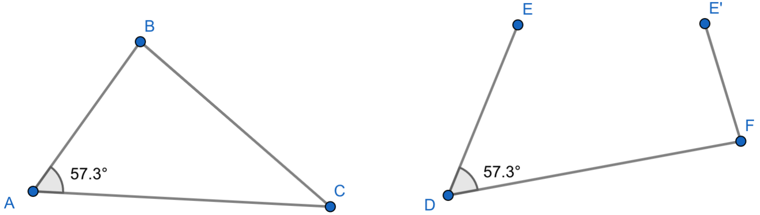


https://www.geogebra.org/geometry/apkxfb5y

Congruence and Similarity: Proof of SAS

http://www.atheistrepublic.com/sites/default/files/styles/blog-featured-image/public/proof.jpg

Congruence and Similarity: Proof of SAS



Let $\triangle ABC$ and $\triangle DEF$ satisfy SAS with $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle BAC \cong \angle EDF$. To show congruence, superimpose the triangles so that A is on D and $\overline{AB}$ is on $\overline{DE}$.

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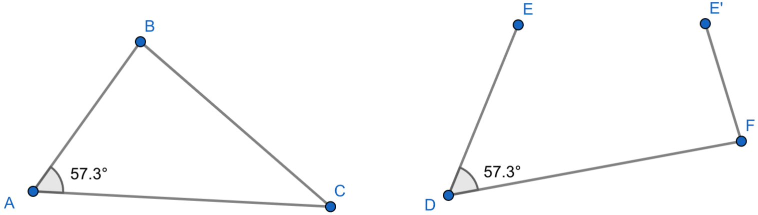

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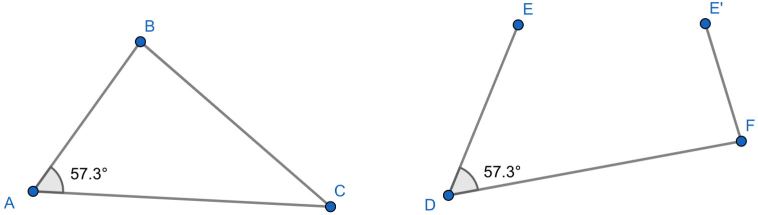

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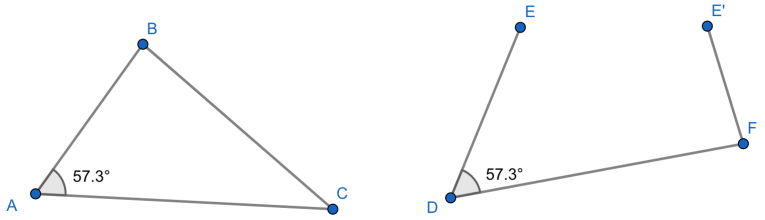

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Linear Transformations of the Plane

Rotation: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ Dilation: $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Projections: y=x line: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ y-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Reflections: y=x line: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Translation: $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$ Others: $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Isometries

isometry: mapping of a metric space onto itself that preserves distance between points
plane: translations, rotations, reflections, and glide reflections
$\rightarrow$ congruence

What preserves shape but not congruence from among dilation, shear, projection?

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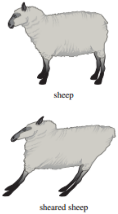
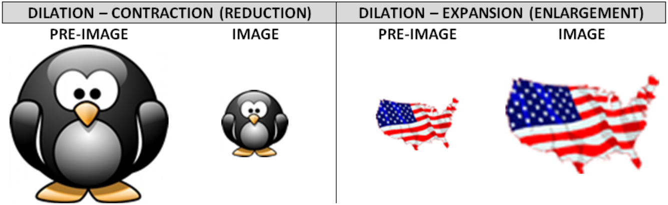


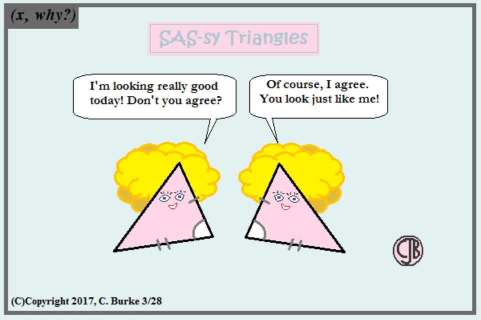


Linear Algebra and Its Applications by David C. Lay,

`http://www.geometrycommoncore.com/content/unit2/gsrt1/teachernotes1.html`

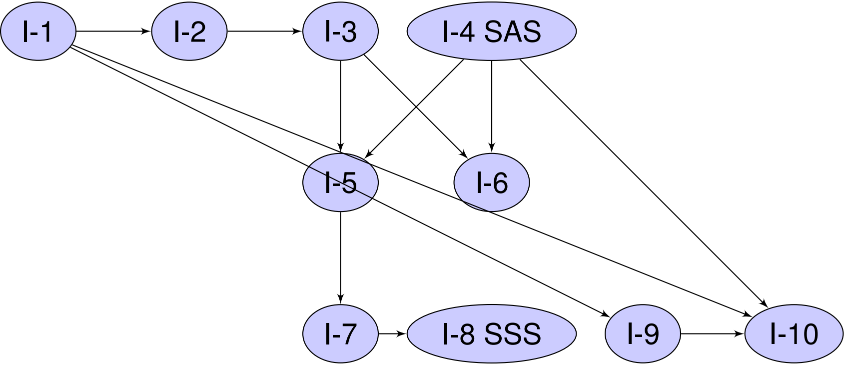
SAS in Other Contexts

- transformations
  https://www.geogebra.org/m/twa9ewvg
- analytic/metric: Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos C$
- other axiom systems: SAS triangle congruence is a postulate not a proposition, or it is proven from one of the other congruence theorems



Propositions, Assumptions and Applications

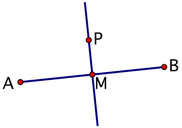
Proof Considerations: I can write rigorous proofs in geometry, identify underlying assumptions, and understand limitations and applications.



SAS Application: Perpendicular Bisectors Equidistance

Sketch of Proof: Given a line segment.

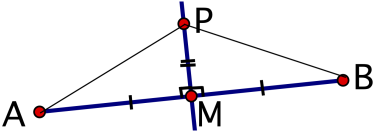
- Proposition 10 constructs midpoint
- Proposition 11 lets us construct a perpendicular through a point on a line creating two right angles
- Select any point on the perpendicular bisector



- Now $\overline{PM} \cong \overline{PM}$ by CN 4 and $\overline{AM} \cong \overline{MB}$ by midpoint
- By Postulate 4, all right angles are equal so if we use postulate 1 to construct $\overline{AP}$ and $\overline{PB}$, then we have SAS in $\triangle PMA$ and $\triangle PMB$.
- By Proposition 4, $\triangle PMA \cong \triangle PMB$ so $\overline{AP} \cong \overline{PB}$

SAS Application: Perpendicular Bisectors Equidistance

Proof: Given $\overline{AB}$ we will show that we can construct a perpendicular bisector which has the property that any point on it is equidistant to the endpoints. Construct the midpoint M by I-10. Apply I-11 to construct the perpendicular bisector through M to $\overline{AB}$. Let P be a point on the perpendicular bisector. Construct $\overline{AM}$, $\overline{MB}$, $\overline{PM}$, $\overline{AP}$ and $\overline{PB}$ by postulate 1. To show that $\overline{AP}$ and $\overline{PB}$ are congruent, we will show that $\triangle PMA$ and $\triangle PMB$ are congruent by SAS I-4. Now $\overline{PM} \cong \overline{PM}$ by CN 4 and $\overline{AM} \cong \overline{MB}$ since M is the midpoint of $\overline{AB}$. Also, $\angle AMP \cong \angle BMP$ by postulate 4 since $\overline{PM}$ is on the perpendicular bisector we constructed. So we have satisfied the assumptions of I-4 SAS. Thus the conclusion of I-4 applies here that $\triangle PMA \cong \triangle PMB$. So $\overline{AP} \cong \overline{PB}$, as desired.

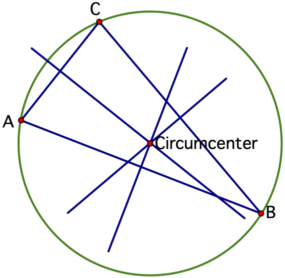


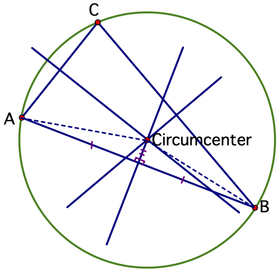
Extension: Circumcenter

- Look at the intersection of 2 perpendicular bisectors of $\triangle ABC$, say the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$.

Extension: Circumcenter

- Look at the intersection of 2 perpendicular bisectors of $\triangle ABC$, say the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$.
- It must be equidistant to all 3 vertices of the triangle.
- The 3rd perpendicular bisector contains all the points that are equidistant to 2 of the vertices, so the intersection too.
- All 3 perpendicular bisectors intersect at the circumcenter.





AA Similarity and SAS Proportionality

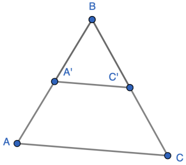
AA: If two angles of a triangle are congruent then the triangles are similar.

SAS Congruence: If two sides of a triangle and the included angle are congruent then the triangles are congruent

SAS Proportionality: If two sides are proportional with the same constant of proportionality and the included angle is congruent then the triangles are similar with the same proportion

Prove: The segment joining the midpoints of two sides of a triangle is half the length of the third

- Create $\triangle ABC$.
- Under Transform, select Dilate from point. Select the triangle, B as the center of dilation, and .5 as the dilation factor.



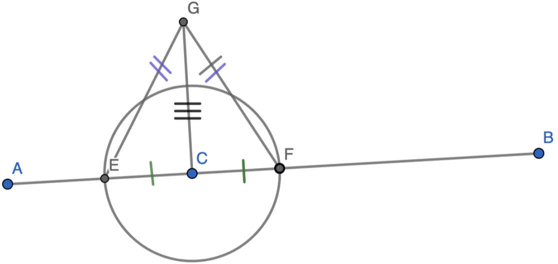
Application of SAS Proportionality

SAS Proportionality: If two sides are proportional with the same constant of proportionality and the included angle is congruent then the triangles are similar with the same proportion

Proof: Let $\triangle ABC$ with A' the midpoint of $\overline{AB}$ and C' the midpoint of $\overline{BC}$ and $\overline{A'C'}$ the segment joining the midpoints all be given. We will show $\overline{A'C'}$ is half the length of $\overline{AC}$. Look at a dilation by $\frac{1}{2}$ from $\triangle ABC$ to $\triangle A'BC'$. By definition of midpoints, $\frac{1}{2}AB = A'B$ and $\frac{1}{2}BC = BC'$. By CN4, $\angle ABC \cong A'BC'$, so $\triangle ABC$ and $\triangle A'BC'$ satisfy the conditions of SAS proportionality, where two sides are proportional and their included angle is congruent. Thus the triangles are similar with the same constant of proportionality, and so $\frac{1}{2}AC = A'C'$. Q.E.D.

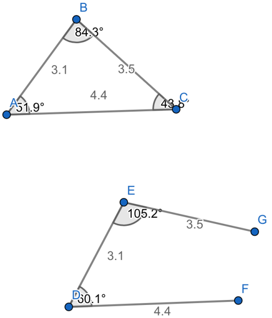
SSS Applied in Euclid's Elements I-11

Part 2 of Proof: To show that $\overline{GC}$ is perpendicular to $\overline{AB}$, compare triangles ECG and FCG, after completing the segments $\overline{EG}$ and $\overline{FG}$ by postulate 1. Notice $CE = CF$ by Def 15 as they are radii of the circle centered at C. In addition $CG = CG$ by CN4. Also, $EG = EF = FG$ because they are constructed using I-1, the equilateral triangle. So $\triangle ECG \cong \triangle FCG$ by SSS congruence in I-8. Thus, $\angle GCE \cong \angle GCF$. By definition 10 the angles must be right.



Congruence and Similarity: Euclidean SSS

- SSS: The three sides of the broken triangle are the same length as the corresponding sides in triangle ABC. How many triangles can we create? Must they be congruent? If not, must they be similar? Why or why not?



https://www.geogebra.org/geometry/bw4eagnh

Congruence and Similarity: Proof of SSS

- Superimpose
- I-7
- CN 4

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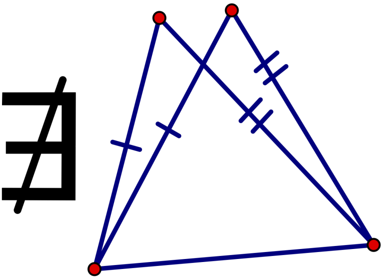

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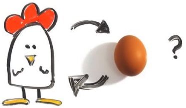
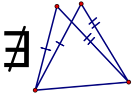
- analytic/metric: Law of Cosines: solve for C in $c^2 = a^2 + b^2 - 2ab\cos C$
- other axiom systems: SSS is a postulate not a proposition and is used to prove SAS
- Hadamard's argument:
  - Move triangle so that $\overline{BC}$ sits on $\overline{B'C'}$, both triangles sit on the same side, and assume for contradiction that A is not on A'
  - B is the same distance from A and A' so it lies on the perpendicular bisector of $\overline{AA'}$. C is similar, so $\overline{BC}$ is the perpendicular bisector.

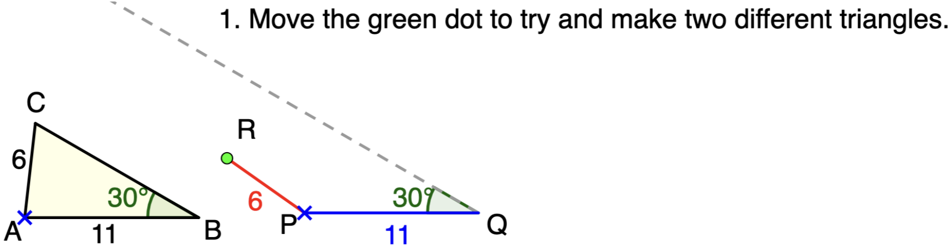




www.brainpickings.org/2013/02/01/which-came-first-the-chicken-or-the-egg/

Congruence and Similarity: Euclidean SSA

- SSA: Two sides of the broken triangle DEF are the same length as the corresponding sides in $\triangle ABC$ ($\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$) . An opposite angle is congruent ($\angle ACB \cong \angle DFE$). How many triangles can we create? Must they be congruent? If not, must they be similar? Why or why not?



https://www.geogebra.org/m/g2cwsz4r

Congruence and Similarity: Proof of (not) SSA