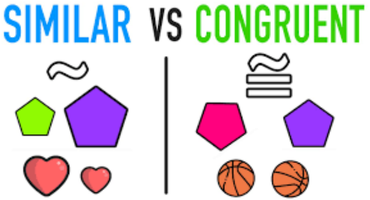
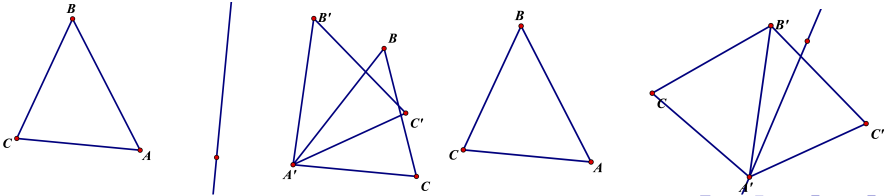
Applications of Congruence and Similarity

- For triangles, SSS and more are lists of possible design choices in Euclidean geometry of the plane.
- Two isometries which agree on three non-collinear points are the same on all points (via SSS)
- Every isometry in the plane is a product of at most three reflections (via SSS)
- More generally, in other geometries, or for more points, like for CAD or CAGD programming, when are design choices complete or when do we need more choices?
- Many additional real-life applications



SSS $\Longrightarrow$ Product of at Most 3 Reflections

Let i be an isometry and look at 3 non-collinear points in the plane A, B, C. Define $i(A) = A'$, $i(B) = B'$, $i(C) = C'$. We'll show we need at most three reflections to represent i. If $A \neq A'$ then reflect in the perpendicular bisector of $\overline{AA'}$ to send A to A'. Next, after the first reflection, if $B \neq B'$ then reflect in the perpendicular bisector of $\overline{BB'}$ to send B to B'. Since an isometry i is distance preserving by definition, then $\overline{AB} = \overline{A'B'}$, $\overline{BC} = \overline{B'C'}$ and $\overline{CA} = \overline{C'A'}$, so the two triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy SSS and hence must be congruent. Thus A must still coincide with A' after the second reflection. If, after the second reflection, $C \neq C'$ reflect across $\overline{A'B'}$. Two isometries which agree on three non-collinear points are the same on all points, so i is the product of at most 3 reflections.



Euclidean Quadrilateral Design Choices

Consider what is the smallest amount of information we need to determine that quadrilaterals are congruent or similar?

Euclidean Quadrilateral Design Choices

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- SSSS in quadrilaterals
  `https://www.geogebra.org/m/kzz5dgsa`
- AA triangles
  `https://www.geogebra.org/m/nhmyevwn`
- AAA and AAAA quadrilaterals
  `https://www.geogebra.org/m/bgejumzb`

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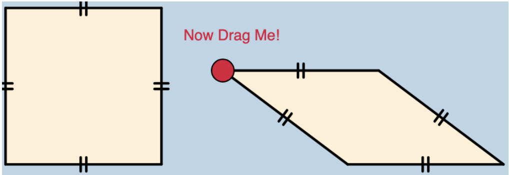
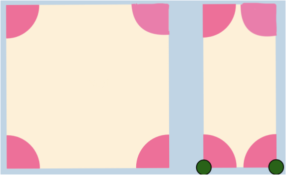
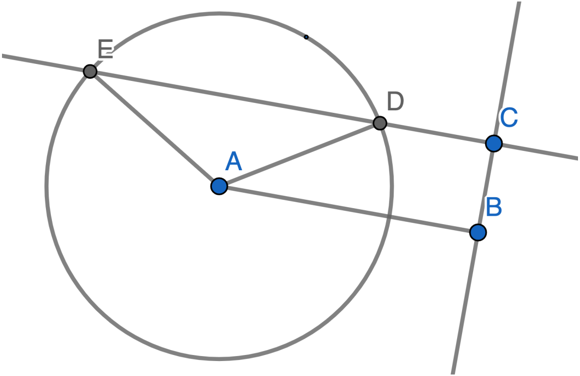




Adapted from Tim Brzezinski, from above links

SSASA Similarity for a Euclidean Quadrilateral?

Consider DABC and EABC



SSASA Similarity for a Euclidean Quadrilateral?

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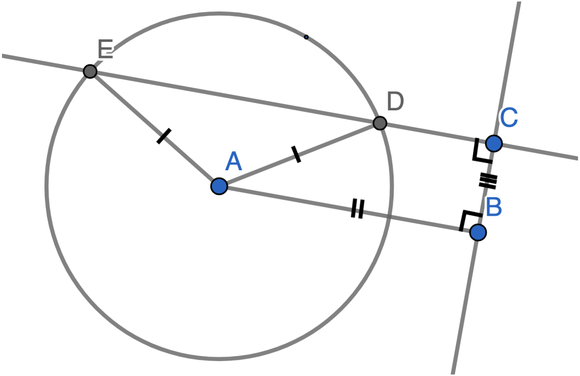
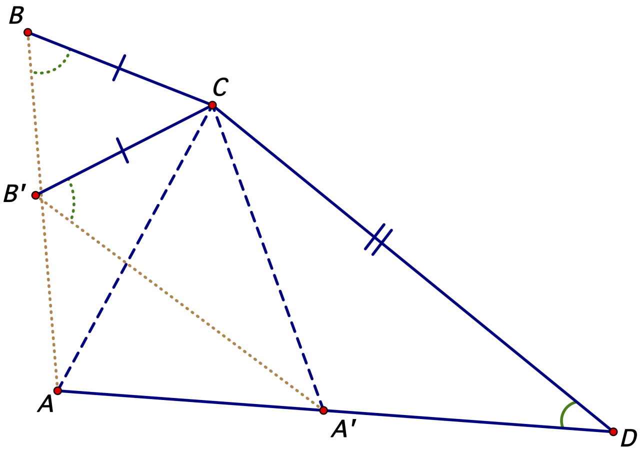


SSASA is not a similarity theorem
implications: SASA, SSSA, SSAA, SSAS
reverse direction from C: ASASS
implications: ASAS, ASSS, SASS, AASS

ASSA Similarity for a Euclidean Quadrilateral?

ASSA Similarity for a Euclidean Quadrilateral?



Triangle ABC rotated about C gives B′

Triangle ABC rotated about C until A intersects AD at A′

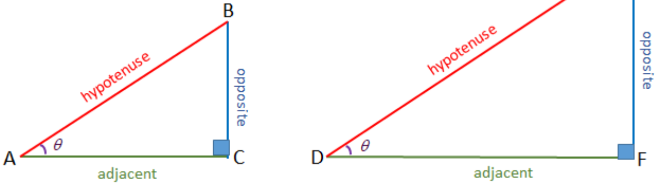
ASSA is not a similarity theorem

AA Similarity $\Longrightarrow$ Trigonometric Functions

Trigonometry and Similar Triangles

If two triangles are similar, then their corresponding sides are proportional and their corresponding angles are congruent.

This can lead to definitions of trigonometric ratios for acute angles in right triangles.



$$\frac{BC}{BA} = \frac{EF}{ED}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

SOH

$$\frac{AC}{AB} = \frac{DF}{DE}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

CAH

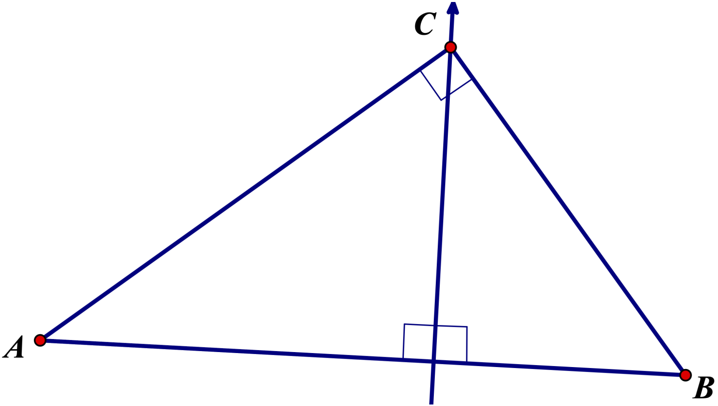
$$\frac{BC}{CA} = \frac{EF}{DF}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

TOA

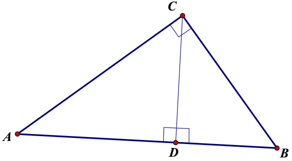
https://www.onlinemathlearning.com/similarity-trig-ratios-hsg-srt6.html

Which Triangles are Similar?



AA Similarity $\implies$ Pythagorean Theorem

Given a right $\triangle ABC$ with hypotenuse $\overline{AB}$ and shortest side $\overline{BC}$, construct a perpendicular from $\overline{AB}$ through C by I-12 and label the intersection as D. Now $\triangle ABC \sim \triangle CBD$ by AA as all right angles are congruent by Postulate 4, $\angle ACB \cong CDB$, and $\angle CBA \cong CBD$ by CN 4. By similarity of $\triangle ABC$ and $\triangle CBD$, since the ratio of the hypotenuse to the shorter base side must be the same, $\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{BC}}{\overline{BD}}$. Cross multiply to see $\overline{AB}\,\overline{BD} = \overline{BC}^2$. Also, $\triangle ABC \sim \triangle ACD$ by AA as $\angle ACB \cong CDA$ by Postulate 4 and $\angle CAB \cong CAD$ by CN 4. $\triangle ABC \sim \triangle ACD$ so the ratio of the hypotenuse to the longer base side must be the same, $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AC}}{\overline{AD}}$. Cross multiply: $\overline{AB}\,\overline{AD} = \overline{AC}^2$. By CN 2, $\overline{AB}\,\overline{BD} + \overline{AB}\,\overline{AD} = \overline{BC}^2 + \overline{AC}^2$. Factor $\overline{AB}$ so that $\overline{AB}(\overline{BD} + \overline{AD}) = \overline{AB}^2$ by CN 4. So $\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$ by CN 1.



Geometric Modeling Using Similarity

geometric similarity is common in deriving and testing physical and biological relationships

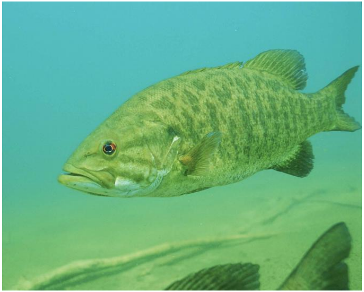


Image 1 public domain, 2.`https://www.atlantisgozo.com/bothus-podas-wide-eyed-flounder/`

a species of bass? $\frac{\text{length}}{\text{height}}$ humans? $\frac{\text{armspan}}{\text{height}}$ i.e. $l \propto h$.

volume?

Geometric Modeling Using Similarity

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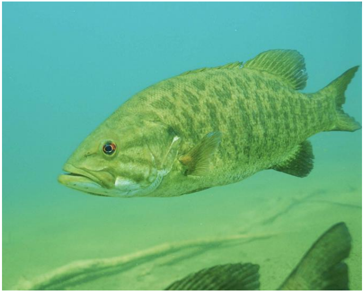

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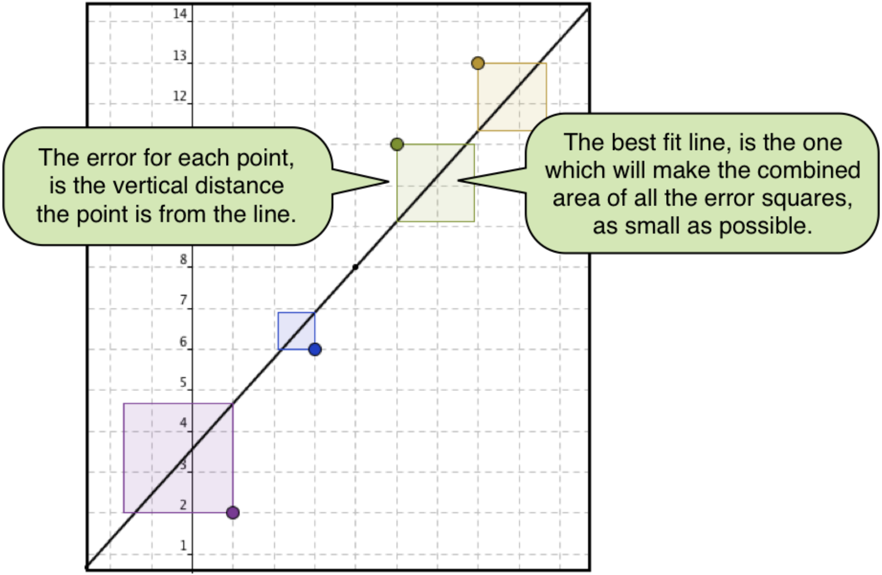
a species of bass? $\frac{\text{length}}{\text{height}}$ humans? $\frac{\text{armspan}}{\text{height}}$ i.e. $l \propto h$.

volume? $v \propto l^3$

surface area? $a \propto l^2$

weight = volume $\times$ average weight density,

so if that density is constant, then $w \propto l^3$



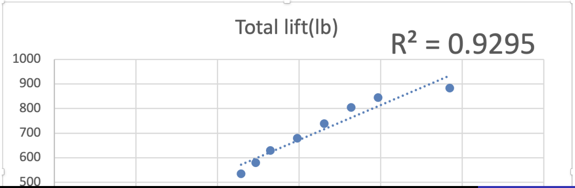
http://math.maine121.org/welcome/chapter-5/

Geometric Modeling Using Similarity

winners of the 1976 Olympics by their weight lifting class

- $\frac{\text{lift}}{\text{weight}} =$ constant slope, i.e. lift $\propto$ weight?
- R^2 measures the y-values distances: sum of squares as variation in the dependent variable explained by linearity. the closer to 100% the stronger the relationship

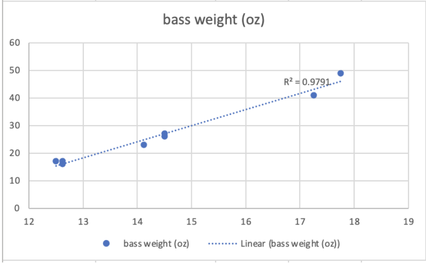
Class	Max weight(lb)	Total lift(lb)
Flyweight	114.5	534.6
Bantamweight	123.5	578.7
Featherweight	132.5	628.3
Leightweight	149	677.9
Middleweight	165.5	738.5
Ligh heavyweight	182	804.7
Middle heavyweight	198.5	843.3
Heavyweight	242.5	881.8

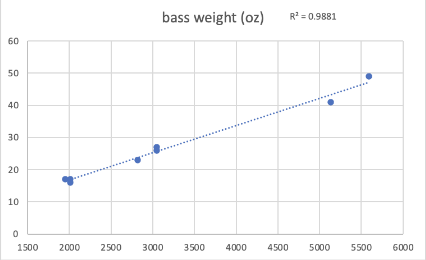


Geometric Modeling Using Similarity

basssimilarity.xlsx bass length (in) and bass weight (oz)

bass length (in)	bass length^3 (in^3)	bass weight (oz)
14.5	3048.625	27
12.5	1953.125	17
17.25	5132.953125	41
14.5	3048.625	26
12.625	2012.306641	17
17.75	5592.359375	49
14.125	2818.158203	23
12.625	2012.306641	16

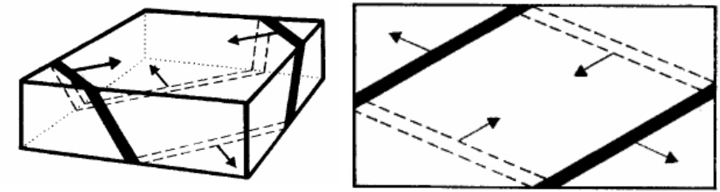




bass length $R^2 = .9791$

bass length^3 $R^2 = .9881$

Sliding a Ribbon Off a Box Using Similarity



The Geometric Viewpoint: a Survey of Geometries by Thomas Sibley