Worksheet on Hyperbolic Geometry Part I

Dr. Sarah's MAT 3610: Introduction to Geometry

goals:

- IGS Exploration
  I can use Interactive Geometry Software (IGS) to discover relationships and demonstrate they seem to apply in a wide variety of examples.
- Proof Considerations
  I can write rigorous proofs in geometry, identify underlying assumptions, and understand limitations and applications.
- Geometric Perspectives
  I can compare and contrast multiple geometric perspectives.

Shortest Distance Paths

1. Open up `https://www.geogebra.org/m/d845j9a3`. We have two paths between A and B. Notice that the path through C looks like the intrinsically straight paths that cut angels and demons in half in Escher's *Circle Limit IV: Heaven and Hell*. Sketch a picture of the two paths.

2. Which path is shorter using the hyperbolic metric in this model? The path from A to B through C or through D?

3. Drag B up toward A (but keep the configuration the same with C to the left of D and B below them). What happens to the difference between the distances?

4. Open up `https://www.geogebra.org/m/R5e9AggU`. Under the first wrench tool, find the **Hyperbolic Segment** tool. Create a hyperbolic segment. Then select the (usual) move symbol (pointer) and drag the endpoints around but inside the disc. Can you obtain the different types of paths that Escher represented as cutting angels and demons in half? Sketch pictures here:

Sum of the Angles

5. What was the sum of the angles at the three points we explored in Escher's *Circle Limit IV: Heaven and Hell*?

6. How did we compute the sum of the angles there?

7. Open up `https://www.geogebra.org/m/svywsx3r` to explore the sum of the angles in hyperbolic geometry more generally. Drag A, B, and C but keep the same configuration (so that you don't end up with an exterior angle measurement—i.e. don't let A cross $\overline{BC}$ or its extension, and similar

for the other points) and keep them all inside the disc. How small can the sum of the angles get in this IGS exploration? What kind of triangle must we form in this model to get a small angle sum?

8. How large can the sum of the angles get in this IGS? What kind of triangle must we form in this model to get a large angle sum?

9. Sketch pictures that illustrate your prior two responses.

Euclid's 5th Postulate

10. Write down the statement of the 5th postulate from *Euclid's Elements* Book I

11. Open `https://www.geogebra.org/m/qmbmeas9`. Do the intrinsically straight paths in the sketch seem to satisfy the assumptions/conditions as well as the conclusion of Euclid's 5th Postulate?

12. Sketch a picture and identify the components in Euclid's 5th.

13. Drag point E so that the assumptions/conditions of Euclid's 5th Postulate still show as holding. Does the conclusion always appear to continue to hold?

14. Sketch a picture and identify the components in Euclid's 5th.

15. Based on #12–15, is Euclid's 5th postulate true in hyperbolic geometry? A postulate holds in a geometry if it holds in all cases.

16. Do the assumptions of Euclid's 5th postulate ever lead to the conclusion in hyperbolic geometry?

Playfair's Postulate

17. Playfair's axiom says: given a line and a point not on it, exactly one line parallel to the given line can be drawn through the point. To show we can create a path that doesn't intersect, we'll create

a hyperbolic sketch that follows along with the Euclidean proof, so open up a new version of `https://www.geogebra.org/m/R5e9AggU`.

–Under the first wrench, use the **Hyperbolic Line** to create a line going through two points. Notice that the program will show four points, two in the disc A and B and two on the boundary at infinity.

–Under the second wrench, use the **Hyperbolic Drop Perpendicular** and select A, B, and a point off the line through them (E). Notice that you will have a perpendicular to $\overline{AB}$.

–Next, under the second wrench, select **Hyperbolic Perpendicular at Point** and choose E and any another point on the perpendicular aside from the intersection with $\overline{AB}$. You have created the perpendicular to the perpendicular, which never intersects $\overline{AB}$, so it is parallel.

This part of the Euclidean proof only required up to I-27, which did not require Euclid's 5th postulate, so it is not surprising that the construction still works in this model of hyperbolic geometry. Sketch a diagram of your construction.

18. Use the **Hyperbolic Angle** under the first wrench to measure the angles and verify that they are right angles (you can add the points you need using the regular point tool). You may have to reverse the order if you obtain the exterior angle rather than the interior angle.

19. Connect E with B with a **Hyperbolic Segment** under the first wrench and then measure the alternate interior angles of the parallels cut by $\overline{EB}$ using **Hyperbolic Angle**. Be careful to measure the alternate interior angles of $\overline{EB}$ rather than the 1st perpendicular. You may have to reverse the order if you obtain the exterior angle rather than the interior angle. Do the alternate interior angles seem to be equal?

20. Sketch a picture that illustrates your response and identify the components.

21. Review I-29. Does the picture indicate that it fails?

SMSG Postulate 16

22. Write down the form of the parallel postulate given in the SMSG Axioms as SMSG Postulate 16?

23. Is SMSG Postulate 16 true on the sphere? Sketch a related picture.

24. Is SMSG Postulate 16 true in Euclidean geometry? Sketch a related picture.

25. Is SMSG Postulate 16 true in hyperbolic geometry? Sketch a related picture.

What Goes Wrong with the Euclidean Proof of the Sum of the Angles

26. Review our Euclidean proof that the sum of the angles in a triangle is 180° (I-32). What goes wrong in the Euclidean proof for hyperbolic geometry? Use what we did in the explorations above as you analyze the proof in order to help you answer this question.

27. Show a hyperbolic sketch as well as each of the steps in the proof up to and including the very first place that the proof fails, and annotate what goes wrong.

Hyperbolic Parallel Axiom

28. The Hyperbolic Parallel Axiom states that if l is a hyperbolic line and P is a point not on l, then there exist exactly two noncollinear hyperbolic halflines which do not intersect l and such that a third hyperbolic halfline intersects l if and only if it is between the other two (and doesn't intersect it otherwise). Try to make sense of this axiom by creating a hyperbolic sketch that illustrates it in `https://www.geogebra.org/m/R5e9AggU`. Be sure to use the **Hyperbolic Line** tool to create the line l. Use the usual point tool to create a point P off the line (it might be called E). Next select the **Hyperbolic Segment** tool to create a hyperbolic segment through P that **does intersect** l—put your second endpoint on the other side of l. Sketch a picture.

29. Move the endpoint (it might be called F) close to the boundary but stay inside the disk. Drag it all the way around to see when the segment will intersect l and when it won't. Try to find the halflines that are mentioned and sketch a picture.