Worksheet on Hyperbolic Geometry Part II

Dr. Sarah's MAT 3610: Introduction to Geometry

goals:

- IGS Exploration
  I can use Interactive Geometry Software (IGS) to discover relationships and demonstrate they seem to apply in a wide variety of examples.
- Proof Considerations
  I can write rigorous proofs in geometry, identify underlying assumptions, and understand limitations and applications.
- Geometric Perspectives
  I can compare and contrast multiple geometric perspectives.

Pythagorean theorem

1. Open up `https://www.geogebra.org/m/jxejcdy5` to explore the Pythagorean theorem in hyperbolic geometry. Does the Pythagorean theorem appear to hold in this model of hyperbolic geometry?

2. Drag the vertices but keep them all inside the disc and keep the configuration the same. What seems to happen to the Pythagorean theorem in small triangles?

3. Sketch pictures that relate and identify geometric components.

4. Open up `https://www.geogebra.org/m/R5e9AggU`.

   –under the first wrench tool, find the **Hyperbolic Segment** tool and create a segment $\overline{AB}$

   –under the second wrench, apply the **Hyperbolic Perpendicular At Point** by selecting A first and then B. You should now have a hyperbolic perpendicular through A to $\overline{AB}$.

   –under the first wrench, use the **Hyperbolic Circle** to select A first and then B. This is a hyperbolic circle with center A and radius $\overline{AB}$.

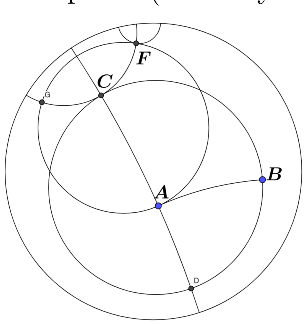
   –under the point tool, choose **Intersect** and select the perpendicular and the circle.

   –next apply the **Hyperbolic Perpendicular At Point** again, but now select C first and then A.

   –apply the **Hyperbolic Circle** and choose C and then A

   –**Intersect** the new circle with the latest perpendicular.

   –Select the intersection point closest to B and use **Hyperbolic Perpendicular At Point** to select that point (F in my diagram) and C.



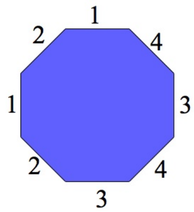
We have three equal width segments ($\overline{AB}$, $\overline{AC}$, and $\overline{CF}$ in my picture) and three 90° angles. In Euclidean geometry, the figure would have already closed up to form a square. Here in this model does this form a square that encloses four 90° angles and four equal sides?

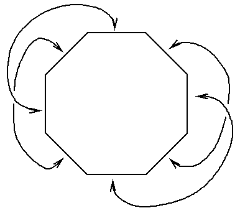
5. Drag A and B—can we form a square as we move A and B around?
6. As a review, sketch a diagram from the Euclidean proof of the Pythagorean theorem from *Euclid's Elements.*
7. As a review, sketch a diagram from the Zhou Bi Suan Jing or Chou Pei Suan Ching proof and puzzle of the Euclidean Pythagorean theorem.
8. Based on your explorations above, what goes wrong with these Euclidean proofs in hyperbolic geometry?

Another Model of Hyperbolic Geometry

9. Search the web to find the measure of one interior angle of a flat octagon. What is it?
10. Why is that the interior angle? Subdivide the octagon into triangles that emanate from one vertex. Write a proof and identify assumptions—you should use that the sum of the angles in a flat triangle is 180°—and also create an accompanying sketch.

11. On an octagon we glue the side with a number on it with the side that has the same number on it. It is an exercise in visualization skills to see that the resulting figure is a 2-holed donut:

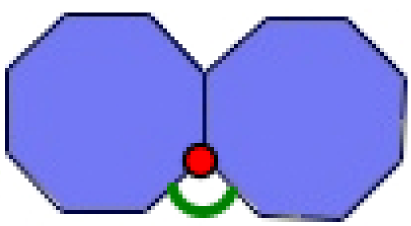





Images: Sascha Rogmann http://www.rogmann.org/math/tori/torus2en.html

To understand why Euclidean geometry does not apply to the 2-holed donut, we can look to see whether octagons will tile the plane or not. So we would like to know whether we can take a certain number of octagons (instead of angels and demons like Escher used in the distorted hyperbolic *Heaven and Hell* work...) and put them together around a vertex in order to form 360°. First, if we put two of them together, and want to understand how much angle they take up, we can double a single interior angle of a flat octagon—since they each take up the same amount of space. So double your angle from your response in #10:

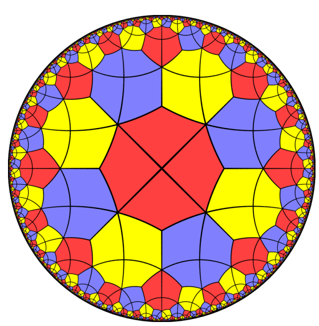
12. How much of 360° is left over at the red point when two octagons are placed side by side—the leftover angle is indicated by the green arc (360°− response from #12)?



13. What happens if we try and place three octagons together at a vertex? Could they fit into 360°?

14. We can create a 2-holed donut by using distorted octagons with 45° interior angles that fit together to tile hyperbolic space. Eight of these glue together like in Escher's work to form 360° at a vertex and so they tile the space ($45 \times 8 = 360$). Now we understand some of the issues that Escher faced, why his *Heaven and Hell* work looked like it did, and why these spaces are not flat—in hyperbolic geometry sides bow in to be able to fit more together.

crocheted hyperbolic octagon



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Circle Limit 4: Heaven and Hell by M.C. Escher, 1960

SAS in various geometries

15. Review the proof of proposition 4 in *Euclid's Elements* Book I via `https://mathcs.clarku.edu/~djoyce/java/elements/bookI/propI4.html`.

16. Where does the Euclidean proof first fail in taxicab geometry? Show a taxicab sketch and write down the first part of the proof that fails.

17. Where does the Euclidean proof first fail in spherical geometry? Show a spherical sketch and write down the first part of the proof that fails.

18. Is I-4 true for hyperbolic triangles? Show a hyperbolic sketch and write down each of the steps in the Euclidean proof and annotate whether each step works in hyperbolic geometry or not.