## Perspective Drawing and Projective Geometry



Figure 8. Albrecht Dürer, From "Unterweisung der Messung."
Woodeut 1525

- Albrecht Dürer (1471-1528), Leonardo Da Vinci (1452-1519), Brook Taylor (1685-1731)
- Industrial Revolution
- Properties shared by 2 perspective views of same scene?
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Evaluate the following arguments:
a) For Playfair's $\rightarrow$ cuts

Assume a straight line intersects one of two parallel lines. If, for contradiction, it doesn't meet the second parallel, then we would have 2 parallels through the intersection, contradicting Playfair's, so cuts holds.
b) For cuts $\rightarrow$ Playfair's

Create the parallel $p$ through $P$ that is the perpendicular to the perpendicular of $I$ (using I-12 and I-11 and I-16). Any other line through $A$ cuts $p$ so by cuts, it also has to cut $l$. Thus Playfair's holds.

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In hyperbolic geometry, as distance between $m$ and any line intersecting it grows, there are examples where it never overtakes the distance between $m$ and $I$, which is how we get infinitely many parallels. Sorry Proclus. Good try though!


## Desargues' Theorem Proof and Applications



Image 2: http://1.bp.blogspot.com/-OiQ_BvieIp4/T4-0nWOvZZI/AAAAAAAACD8/tiTdm6tTrGU/
s1600/checkerboardPersp-01.png
Lift $P$ and $A B C$. If $A B C$ and $A^{\prime} B^{\prime} C^{\prime}$ are in planes that are not parallel, then the planes intersect in a

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Lift $P$ and $A B C$. If $A B C$ and $A^{\prime} B^{\prime} C^{\prime}$ are in planes that are not parallel, then the planes intersect in a line.

## Desargues' Theorem Proof and Applications



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Lift $P$ and $A B C$. If $A B C$ and $A^{\prime} B^{\prime} C^{\prime}$ are in planes that are not parallel, then the planes intersect in a line. Now a side of $A B C$ is on the one plane and the corresponding side of $A^{\prime} B^{\prime} C^{\prime}$ the other, so the intersections of the corresponding sides of the triangles are in both planes and thus on this line. Project.

## What happens if we drag $A B$ parallel to $A^{\prime} B^{\prime}$ ?



- ABint
${ }^{3}{ }^{\text {cint }}$
- Girard Desargues (1591-1661) explored conics as perspective deformations of a circle and the intersection of parallel lines at infinity


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- ABint
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- Girard Desargues (1591-1661) explored conics as perspective deformations of a circle and the intersection of parallel lines at infinity
- Jean-Victor Poncelet (1788-1867) added points and explored properties invariant under projection


## Desargues in Hyperbolic Geometry



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Perspective Drawing and Projective Geometry

## Desargues in Spherical Geometry



## Desargues in Spherical Geometry



## Desargues in Spherical Geometry



- Pairs of "parallels" allow the fluid motion to continue undisturbed https://youtu.be/e2kHrDRXzP4
- AAA fails for degenerate triangles and gives congruence


## What is Projective Geometry? A Sphere Divided

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- Plane: $(x, y) \rightarrow(x+1, y)$



## Surfaces that Locally Look Like the Plane?

- Felix Klein (1849-1925) posed the question in 1890
- In Klein's Erlangen Program, the properties of a space were understood by the transformations that preserved them.
- Heinz Hopf's (1894-1971) rigorous solution was 1925 A complete connected surface which locally looks like the plane is obtained via a quotient by a group of isometries acting without fixed points



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Two points are the same if and only if we can get from one point to the other by a transformation: plane, cylinder, Mobius Band, flat Clifford torus, flat Klein bottle


## Surfaces that Locally Look Like the Sphere: $\mathbb{R} \mathrm{P}^{2}$

- Projective Geometry
$\frac{S^{2}}{\Gamma}$ where $\Gamma=\{$ identity, $(x, y, z) \rightarrow(-x,-y,-z)\}$.



## Projective Geometry: $\mathbb{R} \mathrm{P}^{2}$



- Elegant
- Duality between points and lines
- Conics
- SAS is fixed (although l-16 still fails)


## Hierarchies of Geometries via Transformations

projective transformations
hyperbolic
Euclidean

- Euclidean transformations $\subset$ similarity transformations (includes scalings) $\subset$ projective transformations
- spherical and hyperbolic $\subset$ projective
- smaller the transformation group, the more rigid and more invariants.


## Hierarchies of Geometries via Transformations

## projective transformations

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- spherical and hyperbolic $\subset$ projective
- smaller the transformation group, the more rigid and more invariants.
- Arthur Cayley (1821-1895): "projective geometry is all geometry"

Which do you find most compelling about why Euclid (~ 325-~ 265 BCE) wrote the 5th postulate the way he did?
a) Euclid's 5th might be more of a self-evident truth in nature than Playfair's as it holds in spherical geometry
b) In Prop 31, Euclid constructs a parallel, but he doesn't use the language of uniqueness (there can be only one) in Book 1
c) Euclid was trying to keep the same kind of language as the other postulates
d) It is easier to use Euclid's 5th in the propositions to help prove and support them than it would be in using Playfair's
e) There was no notion of infinity then, so instead of Playfair's which refers to never intersecting, Euclid's 5th gives something constructive about intersection
f) To motivate others to work on parallels and resolve issues he hadn't and better understand the nature of reality
https://www.youtube.com/watch?v=LPET』HhNOVM

