The Beauty of Analytic Hierarchy Process

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Abstract

In our world, making decisions is what human-beings do intentionally or unintentionally to operate throughout every day life. It is always a controversial debate of how one decision is more effective, more efficient, and more important than the others. Furthermore, the process of decision making in a lot of situations would involve with our countless factors such as personal experiences, values, and opinions in considerations of such a decision. Therefore, an applied mathematician in 1980 named Thomas Saaty had formally and intensively developed a technique called Analytic Hierarchy Process or AHP for structuring, measurement, and synthesis in order to organize and analyze complex decisions based on mathematics, philosophy, and psychology.

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1 Introduction

1.1 Prior Experience

1.1.1 MAT 2240: Introduction to Linear Algebra

We explored matrices in 2-D and 3-D. An understanding of eigenvector and eigenvalues corresponding to a square matrix would be essential in order to work on paired comparisons as ratios in AHP with all of the criterions and alternatives.

An eigenvector of a square matrix A is a non-zero vector v that, when the matrix multiplies v, yields the same as when some scalar multiplies v, the scalar multiplier often being denoted by λ. That is: $A\vec{v} = \lambda\vec{v}$ (Because this equation uses post-multiplication by v, it describes a right eigenvector.) The number λ is called the eigenvalue of A corresponding to v.

For the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

We have

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3n \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix}$$

Therefore, the vectors $[1,0,-1]^T$ and $[1,n,1]^T$ are eigenvectors of A corresponding to the eigenvalues 1 and 3 respectively. (Here the symbol T indicates matrix transposition, in this case turning the row vectors into column vectors.)

Moreover, we also have the opportunity to explore the concept of geometric mean. One might misconstrue that the geometric mean is equivalent to the arithmetic mean. However, in the world of mathematics, it they are not the same thing and the geometric mean is not also a simple average neither. It is the nth root of the product of n numbers to indicates the central tendency or typical value of a set of numbers by using the product of their values (as opposed to the arithmetic mean which uses their sum). For instance, the geometric mean of two numbers, say 12 and 3, is just the square root of their product; that is $\sqrt{12 \cdot 3} = 6$. Similarly, the geometric mean of three numbers, say 4, 6, and $\frac{1}{3}$ is the cubic root of their product; that is $\sqrt[3]{4 \cdot 6 \cdot \frac{1}{3}} = 2$

However, for matrices, there would be a hint of complexity adding to the concept of geometric means. For instance, given three assessment matrices:

$$PM1 = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1/4 \\ 1/3 & 4 & 1 \end{bmatrix}, \quad PM2 = \begin{bmatrix} 1 & 3 & 1/2 \\ 1/3 & 1 & 2 \\ 2 & 1/2 & 1 \end{bmatrix}, \quad PM3 = \begin{bmatrix} 1 & 1/2 & 5 \\ 2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}$$

form the assessment matrix $PM = [pm_{i,j}]$ with

$$pm_{i,j} = \text{GeometricMean}(PM1_{i,j}, PM2_{i,j}, PM3_{i,j})$$

$$PM = \begin{bmatrix} 1 & \sqrt[3]{3} & \sqrt[3]{\frac{15}{2}} \\ \sqrt[3]{\frac{1}{3}} & 1 & \sqrt[3]{\frac{3}{2}} \\ \sqrt[3]{\frac{2}{15}} & \sqrt[3]{\frac{2}{3}} & 1 \end{bmatrix} \approx \begin{bmatrix} 1.0 & 1.44 & 1.96 \\ 0.693 & 1.0 & 1.14 \\ 0.511 & 0.872 & 1.0 \end{bmatrix}$$

The critical property of geometric means needed is

$$\text{GeometricMean}(1/x_i) = 1/\,\text{GeometricMean}(x_i).$$

Geometric means ultimately would be an aid to the process of combining multiple assessments in producing aggregate assessments before figuring out the aggregate hierarchy in AHP.

1.1.2 MAT 5340: Operation Research

We explored different types of operation research methodologies in applied mathematics fields and worked intensively in a group project called "Establishing Program Priorities in an Organization Using Analytics". We adapted a method for prioritizing academic programs at Appalachian State University in an organization using Analytic Hierarchy Process (AHP). AHP builds a stratified ordering with relative priority scores from pairwise assessments using common criteria. The attached picture is the poster that our group presented at the 17^{th} *Annual Celebration of Student Research and Creative Endeavors* conference as a result of our research project in Spring 2014.

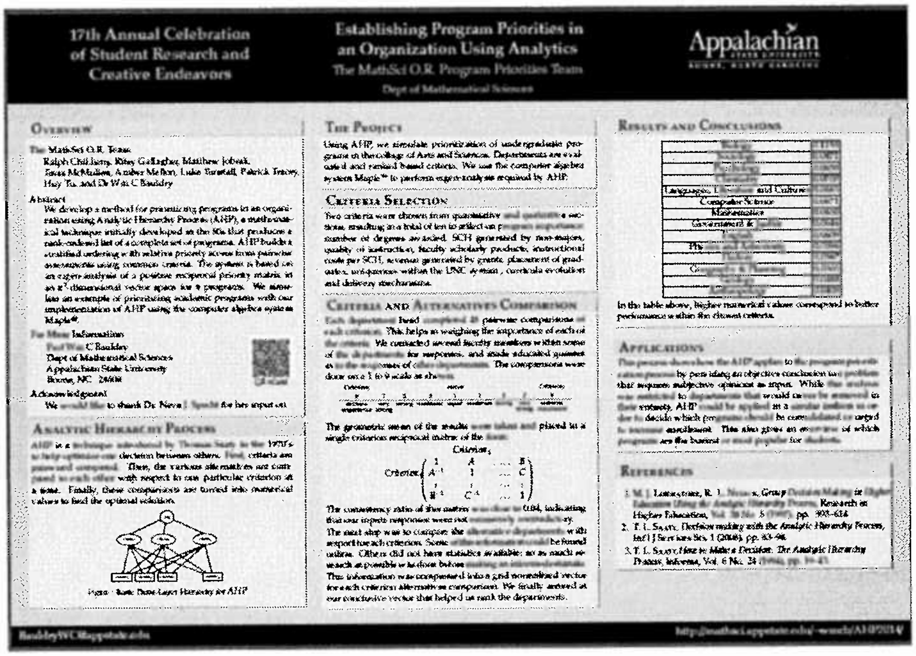

Figure 1: Establishing Program Priorities in an Organization Using Analytics Poster

2 Developments

2.1 Methodology

The AHP has been applied to a wide range of problems and situations due to its simplicity, ease of use, and great flexibility in predicting a situation. Real life applications include total quality management, scarce resource allocation, and picking or choosing among plausible alternatives in the multi-objective environment. The AHP can be applied from everyday life decision makings to different decision situations in professional fields of education, politics, management, social life, business, healthcare, sports, and science. It has particular application in resolution of choice, priority determination, and group decision making in a multicriteria environment. [2]

As an individual starts dwelling into the beauty of mathematics, they would start realizing that mathematics does not always have one correct answer and there might be multiple ones depending on the process of finding and researching it.

Similarly, with the consideration of intangible personal experiences, opinions, and values, there is no definite correct or right decision to make in uncontrollable and unprecedented situations in life. The AHP provides a more in-depth understanding analysis representing all the alternative decisions on a standardly quantified scale of comparison. [2] The scale is unique from situation to situation. It assists decision makers to have a better understanding of the problems and relate those decisions to overall goals in order to find one that suit the goals ultimately which could lead to evaluate alternative solutions. [5]

In *Integrated analytic hierarchy process and its applications*, William Ho referenced that Vaidya and Kumar (2006) found 150 articles investigating the AHP method combined with general applications. With it's wide applicability to any situations that requires structuring, measurement, and/or synthesis, the AHP method should be used as a mean to integrate, support, or combine with other techniques and methods in order to maximize the potential of the project and the result[3]. For instance, Forman and Gass pointed out that when analyzing the most efficient amount of servers to employ in a queue situation, AHP method is used as assistance to queueing theory to measure and synthesize all the alternative preferences of waiting/arrival time, cost, customer, server, and system capacity. [2] Therefore, AHP can be integrated with other techniques such as mathematical programming in order to consider not only both qualitative and quantitative factors, but also some real-world resource limitations. This approach, regarded as the integrated AHP, can definitely make a more realistic and promising decision than the stand-alone AHP. [3]

Aside from the dominance of technical structure and the process of the Analytic Hierarchy Process, the beginning steps of making decisions through selecting the criterias that are important for that decision is the most creative aspect of decision problem. [2] Saaty reminds the decision makers to represent the problem as thoroughly as possible while preserving the sensitivity of the elements, establish the relevant and necessary issues and attributes, beware of the problem's surrounding environment, and indicate the associated participants. [5] Figure 2 shows the standard visual representation of the result of arranging works of all the goals, criterions, and alternatives in a basic three-layer hierarchy.

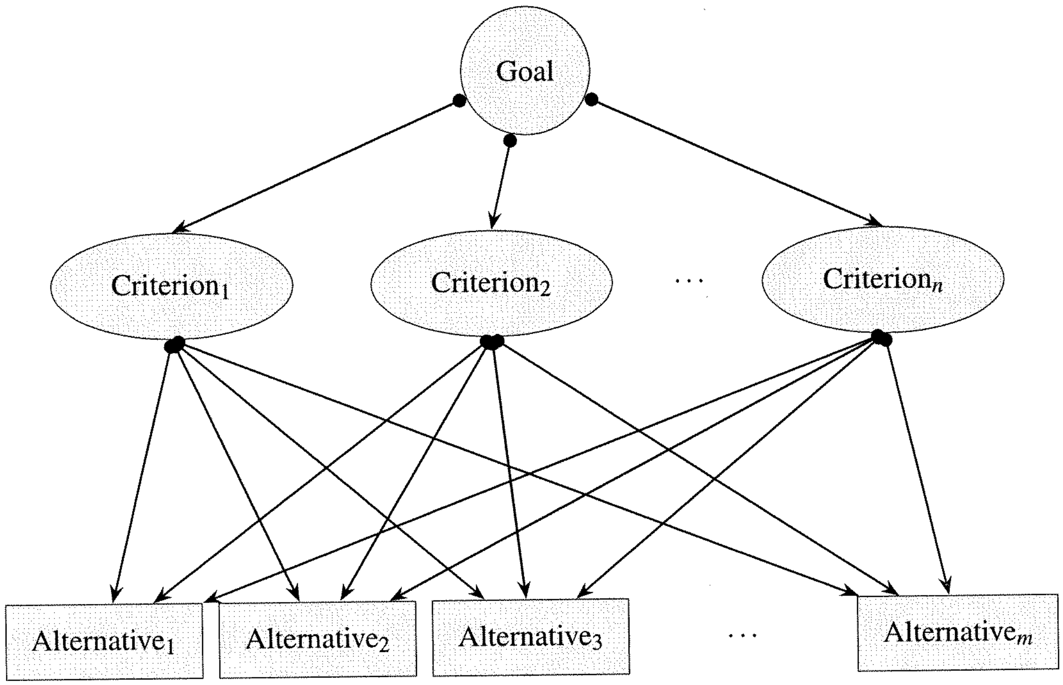

Figure 2: Basic Three-Layer Hierarchy for *AHP*

From the arrangement of goals, criterions, and alternatives, an overall picture of the complex relationship is painted inherently from the decision making situation. Moreover, it assists the decision maker to assess logically whether the issues in each level are of the same significance or weight where they can compare consistently with such homogeneous standard accurately. [5]

So how does one construct the AHP for decision making? Firstly, one must create the decision hierarchy by breaking the decision problem into interrelated decision elements. One must need to figure the objective of the decision problem or what one wants to decide, choose, or pick.[1] Appropriate and specific criteria either subjective criteria or objectives ones, should be listed out for consideration of weighting all of the possible alternatives that one will decide on to satisfy the objective of the decision.[5]

Secondly, one should collect input data with pairwise comparisons of decision elements of necessary matrices. The comparisons would be done on a 1 to 9 as order of magnitude scale as shown in figure 3 between criteria to figure out the most important criterion to the user or a group and then between alternatives in respect to each individual criterion.

Criterion$_1$				*versus*				*Criterion*$_2$
9	7	5	3	1	3	5	7	9
extreme importance	very strong	strong	moderate	equal	moderate	strong	very strong	extreme importance

Figure 3: Basic comparison scale for *AHP*

Let A be a relationship matrix between criteria.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

To achieve the normalized weight or eigenvectors for each criterion, each element of the matrix would be computed as

$$\hat{a}_{i,j} = \frac{a_{i,j}}{\sum\limits_{k=1}^{n} a_{k,j}}$$

in order to achieve normalized criteria matrix A as

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \dots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \dots & \hat{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \dots & \hat{a}_{nn} \end{bmatrix}$$

Next, this step would be where the eigenvalues and eigenvectors become utilitarian for the whole methodology.[4] The user shall compute the eigenvalues through normalize the resule of multiplicating the eigenvector from associated criterions with the combined matrix of all eigenvectors between alternatives in respect to each individual criteria in respective order.[1]

The user shall be able to figure out the weighted matrix that contains the weight factor for each criterion that show which criterion is the most important to the user by using the elementary matrix algebra:

$$\hat{W} = \frac{a_{i,j}}{n} \cdot \hat{A} = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_{n-1} \\ \hat{w}_n \end{bmatrix}$$

where

$$\hat{w} = \frac{1}{n} \sum_{j=1}^{n} a_{k,j}, \qquad k = 1, \ldots, n$$

Similarly, the user can apply the same method to figure out the eigenvector represent the relationship between alternatives to be considered for the decision making process for each respective criterion.[6]

Finally, with the result of the process, the user shall be able to aggregate the relative weights of decision elements to rate the decision alternatives through multiplicating the normalized weighted criteria matrix with the combined matrix from all eigenvectors between alternatives in respect to each individual criteria in respective order.[5] From looking at the final weighted eigenvector, the user can figure out the fittest and most suitable alternative for the user, given the criteria.

In a situation where the decision was made by a committee or a group or a team that is more than one decision maker. The ending process would be the same in finding the relative weights of decision elements, Yet, there would be different results coming from the subjective criteria ranking since everyone have different opinions on things. The geometric mean shall then be used, computed and placed in a single criterion reciprocal matrix of the form [4]

$$\underset{\text{Criterion 1}}{} \overset{\text{Criterion 2}}{\begin{bmatrix} 1 & A & \ldots & B \\ A^{-1} & 1 & \ldots & C \\ \vdots & & \ddots & \vdots \\ B^{-1} & C^{-1} & \ldots & 1 \end{bmatrix}} \xrightarrow[\text{analysis}]{\text{eigen}} \begin{pmatrix} \textit{Criteria} \\ \textit{Comparison} \end{pmatrix} \times \begin{pmatrix} \textit{Priority} \\ \textit{Rankings} \end{pmatrix}$$

2.2 History

2.2.1 1980's

The demands for solving a problem or a controversial issue of how life situations are more than just black or white, right or wrong, and correct or incorrect had increased and inspired the applied mathematician, Saaty, to develop AHP to produce a rank-ordered list of complete sets of complex decisions for the sake of structuring, measurement, and synthesis in order to organize and analyze complex decisions, based on mathematics, philosophy, and psychology.

2.2.2 2000's

In the beginning of 2000's, a DEAHP method was developed by mathematician Ramanathan which is an integrated version of both Data Envelopment Analysis (DEA) and Analytics Hierarchy Process (AHP) methods together. [7] However, mathematicians Ying-Ming Wang and Kwai-Sang Chin discussed in their published paper on 2008 about the new DEA for it's potential applications in priority organization and resolution of choice in the AHP and extends it to the group decision making in a multicriteria environment. [7]

2.3 Recent Scholarly Research

Even when a few of practical and theoretical aspects of Analytic Hierarchy Process are views as controversial, it's acceptance, application, and usage in practice is widely undeniable. Therefore, modern researchers are consistently encourage to use AHP as a mean to incorporate, support, or combine with other techniques while it's elements are constantly being researched, revised, and adapted for more applicable situations. [6]

According to Song-Kyoo Kim, global companies currently pay more attentions and efforts in preserving and improving their innovation through creating business value which can be in many varied forms in order to sustain their existence in today's corporate world. By applying the extensive version of an Analytic Hierarchy Process Expansion framework, the explicit and objective measurement method for innovation method for innovation performances is plausible. The innovation performance measurement factors can be prioritized and descending-order rank list of the performance factors can be made in order to select the most suitable strategies to the companies for the enhancement of their innovativeness. [4]

According to the results of the analysis, output-related performance factors do carry the major weight of determining the innovativeness of the company, which is what most of the current consulting firms do for measuring the innovation of the companies. However, process-related factors such as a distinct process dedicated for innovation still carry a significant amount of weight in the determination consideration as well. [4]

In the research paper titled *The analytic hierarchy process with stochastic judgements* paper, the author investigated an integrated version of AHP through applying the stochastic multicriteria acceptability analysis (SMAA), an inverse-preference method. Even when the AHP has helped eased out the definite requirements with its unique arrangement of factors in a hierarchic structure, the situation can sometime be fickle or difficult to agree on precise pairwise comparisons. Therefore, SMAA-AHP can have the ability to allow flexible modelling of varied kinds of imprecision, uncertainty, information constantly refined during the decision making process, or the missing nature of preference information. Legitimate concerns arose regarding the uncertainty level of information resulting in inconsistency of judgements and alternative outcomes. However, according to the authors, results retrieved from countless simulations indicated that judgements are likely to remain consistent unless uncertainty is severe, but that the presence of uncertainty in almost any degree is sufficient to make the choice of best alternative unclear. [1]

3 Acknowledgements and References

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