SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$
$\vec{F}$ on m?

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $\quad \frac{Mm}{r^2}$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $$\frac{Mm}{r^2}\frac{-\vec{X}}{r} = m\frac{d^2\vec{X}}{dt^2} \text{ so } \frac{d^2\vec{X}}{dt^2} = \frac{-M\vec{X}}{r^3}$$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

SpaceTime-Time: General Relativity

$M = (0, 0, 0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $$\frac{Mm}{r^2}\frac{-\vec{X}}{r} = m\frac{d^2\vec{X}}{dt^2} \text{ so } \frac{d^2\vec{X}}{dt^2} = \frac{-M\vec{X}}{r^3}$$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r}$$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $\quad \dfrac{Mm}{r^2}\dfrac{-\vec{X}}{r} = m\dfrac{d^2\vec{X}}{dt^2}$ so $\dfrac{d^2\vec{X}}{dt^2} = \dfrac{-M\vec{X}}{r^3}$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r} = \frac{M}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} =$$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $$\frac{Mm}{r^2}\frac{-\vec{X}}{r} = m\frac{d^2\vec{X}}{dt^2} \text{ so } \frac{d^2\vec{X}}{dt^2} = \frac{-M\vec{X}}{r^3}$$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\displaystyle\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r} = \frac{M}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r} \text{ so } \frac{\partial\phi}{\partial x} = \frac{Mx}{r^3}$$

$$\frac{\partial^2\phi}{\partial x^2} =$$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $\quad \dfrac{Mm}{r^2}\dfrac{-\vec{X}}{r} = m\dfrac{d^2\vec{X}}{dt^2}$ so $\dfrac{d^2\vec{X}}{dt^2} = \dfrac{-M\vec{X}}{r^3}$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r} = \frac{M}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r} \text{ so } \frac{\partial\phi}{\partial x} = \frac{Mx}{r^3}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{r^3M\cdot 1 - Mx\cdot 3r^2\cdot\frac{x}{r}}{r^6} = \frac{M}{r^5}(r^2 - 3x^2) \text{ similar for } y, z$$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $\quad \dfrac{Mm}{r^2}\dfrac{-\vec{X}}{r} = m\dfrac{d^2\vec{X}}{dt^2}$ so $\dfrac{d^2\vec{X}}{dt^2} = \dfrac{-M\vec{X}}{r^3}$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r} = \frac{M}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r} \text{ so } \frac{\partial\phi}{\partial x} = \frac{Mx}{r^3}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{r^3M\cdot 1 - Mx\cdot 3r^2\cdot\frac{x}{r}}{r^6} = \frac{M}{r^5}(r^2 - 3x^2) \text{ similar for } y, z$$

$\nabla^2\phi = \frac{M}{r^5}(r^2 - 3x^2 + r^2 - 3y^2 + r^2 - 3z^2) =$
$\frac{M}{r^5}(3r^2 - 3x^2 - 3y^2 - 3z^2) = 0 \quad \square$

SpaceTime-Time: General Relativity

$M = (0,0,0)$ and m at $\vec{X}(t) = (x(t), y(t), z(t))$ with $r = |\vec{X}|$

$\vec{F}$ on m? $$\frac{Mm}{r^2}\frac{-\vec{X}}{r} = m\frac{d^2\vec{X}}{dt^2} \text{ so } \frac{d^2\vec{X}}{dt^2} = \frac{-M\vec{X}}{r^3}$$

Define $\phi(r) = \frac{-M}{r}$ as the potential function

Theorem: $\nabla^2\phi = 0$ where $\nabla^2\phi = \frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} + \frac{\partial^2\phi}{dz^2}$ and $r \neq 0$

Proof: First examine $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\frac{-M}{r}}{\partial r} = \frac{M}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r} \text{ so } \frac{\partial\phi}{\partial x} = \frac{Mx}{r^3}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{r^3M\cdot 1 - Mx\cdot 3r^2\cdot\frac{x}{r}}{r^6} = \frac{M}{r^5}(r^2 - 3x^2) \text{ similar for } y, z$$

$\nabla^2\phi = \frac{M}{r^5}(r^2 - 3x^2 + r^2 - 3y^2 + r^2 - 3z^2) =$

$\frac{M}{r^5}(3r^2 - 3x^2 - 3y^2 - 3z^2) = 0 \quad \square$

Corollary:

$$\left(\frac{d^2\vec{x}}{dt^2}, \frac{d^2\vec{y}}{dt^2}, \frac{d^2\vec{z}}{dt^2}\right) = \frac{d^2\vec{X}}{dt^2} = \frac{-M\vec{X}}{r^3} = -\left(\frac{Mx}{r^3}, \frac{My}{r^3}, \frac{Mz}{r^3}\right) = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

SpaceTime-Time: General Relativity

We've been following the 1916 paper: "The Foundation of the General Theory of Relativity"

SpaceTime-Time: General Relativity

We've been following the 1916 paper: "The Foundation of the General Theory of Relativity"

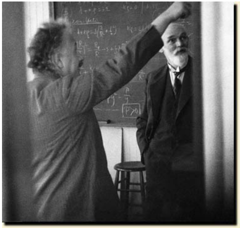
- Einstein replaced the corollary with

$$\frac{d^2x^\lambda}{ds^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

- $\frac{\partial\varphi}{\partial x^i}$ & $\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}$ similar roles: Field equations relate these potential functions to the distribution of matter
- Field equations written in the Christoffel symbols:

$$\frac{\partial\Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda} = 0$$

Solutions for General Relativity

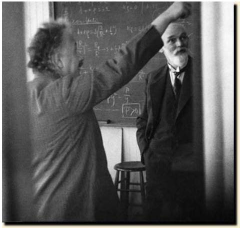


https://history.aip.org/exhibits/einstein/images/ae65.jpg

- 2nd order PDE 16 eqs and 16 unknowns
- Einstein: "Cosmological Considerations in the General Theory of Relativity" (1917)

  It remains now to determine those components of the gravitational potential which define the purely spatial-geometrical relations of our continuum (g_{11}, g_{12}... *the curvature of the required space must be constant...*

Solutions for General Relativity

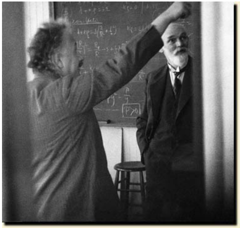


https://history.aip.org/exhibits/einstein/images/ae65.jpg

- 2nd order PDE 16 eqs and 16 unknowns
- Einstein: “Cosmological Considerations in the General Theory of Relativity” (1917)

  It remains now to determine those components of the gravitational potential which define the purely spatial-geometrical relations of our continuum (g_{11}, g_{12}... *the curvature of the required space must be constant... will be a spherical space.*

Solutions for General Relativity



https://history.aip.org/exhibits/einstein/images/ae65.jpg

- 2nd order PDE 16 eqs and 16 unknowns
- Einstein: “Cosmological Considerations in the General Theory of Relativity” (1917)

  It remains now to determine those components of the gravitational potential which define the purely spatial-geometrical relations of our continuum ($g_{11}, g_{12}\ldots$ *the curvature of the required space must be constant... will be a spherical space.*
- Schwarzschild metric (1916), de Sitter: $\mathbb{R} \times S^3$ (1917)
- Einstein & Rosen Schwarzschild $\rightarrow$ wormhole (1935)

"Cosmological Considerations in the General Theory of Relativity" (1917) https://einsteinpapers.press.princeton.edu/vol6-trans/441

our assumption as to the uniformity of distribution of the masses generating the field, it follows that the curvature of the required space must be constant. With this distribution of mass, therefore, the required finite continuum of the x_1, x_2, x_3, with constant x_4, will be a spherical space.

We arrive at such a space, for example, in the following way. We start from a Euclidean space of four dimensions, ξ_1, ξ_2, ξ_3 ξ_4, with a linear element $d\sigma$; let, therefore,

$$d\sigma^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_4^2 \quad . \quad . \quad . \quad (9)$$

In this space we consider the hyper-surface

$$R^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2, \quad . \quad . \quad . \quad (10)$$

where R denotes a constant. The points of this hyper-surface form a three-dimensional continuum, a spherical space of radius of curvature R.

The four-dimensional Euclidean space with which we started serves only for a convenient definition of our hyper-surface. Only those points of the hyper-surface are of interest to us which have metrical properties in agreement with those of physical space with a uniform distribution of matter. For the description of this three-dimensional continuum we may employ the co-ordinates ξ_1, ξ_2, ξ_3 (the projection upon the hyper-plane $\xi_4 = 0$) since, by reason of (10), ξ_4 can be expressed in terms of ξ_1, ξ_2, ξ_3. Eliminating ξ_4 from (9), we obtain for the linear element of the spherical space the expression

$$\left.\begin{aligned} d\sigma^2 &= \gamma_{\mu\nu} d\xi_\mu d\xi_\nu \\ \gamma_{\mu\nu} &= \delta_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{R^2 - \rho^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

where $\delta_{\mu\nu} = 1$, if $\mu = \nu$; $\delta_{\mu\nu} = 0$, if $\mu \neq \nu$, and $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. The co-ordinates chosen are convenient when it is a question of examining the environment of one of the two points $\xi_1 = \xi_2 = \xi_3 = 0$.

Now the linear element of the required four-dimensional space-time universe is also given us. For the potential $g_{\mu\nu}$, both indices of which differ from 4, we have to set

$$g_{\mu\nu} = -\left(\delta_{\mu\nu} + \frac{x_\mu x_\nu}{R^2 - (x_1^2 + x_2^2 + x_3^2)}\right) \quad . \quad (12)$$