Curvatures and Christoffel Symbols of the Poincaré Upper-half Plane Surface

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. $ds^2 = g_{ab}dx^a dx^b$ is the ________________________
2. $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$ are ________________________
3. Discuss these and respond to this last one on `pollev.com/drsarah314`:
   $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ are
   a) metric form
   b) Christoffel symbols
   c) geodesic equations
   d) Riemann curvature tensor or Riemann-Christoffel tensor
   e) other
4. Recall from the 5.4 reading that the Poincaré upper-half plane surface $P = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ with the patch $x(u,v) = (u,v)$ had $g_{11} = E = \dfrac{1}{v^2}$, $g_{12} = g_{21} = F = 0$, and $g_{22} = G = \dfrac{1}{v^2}$. Compute $K = -\dfrac{1}{2\sqrt{EG}}\left(\dfrac{\partial}{\partial v}\left(\dfrac{E_v}{\sqrt{EG}}\right) + \dfrac{\partial}{\partial u}\left(\dfrac{G_u}{\sqrt{EG}}\right)\right)$ and show work. Compare with your group.
5. For the Christoffel symbols and various additional curvatures, there are a lot of computations, and so it is standard to use a computer algebra software system, especially for spacetime where there are even more computations! We'll do that too, but first though, let's get used to Einstein summation notation and more.

   Let's begin with Γ^2_{11} for the Poincaré upper-half plane surface via direct computation from the definition. Anytime there is an upper index that matches the lower index, we sum over all the possible coordinates—1 to 2 for surfaces and 1 to 4 for spacetime. For Γ^2_{11}, we see that $b = 1, c = 1, a = 2$ in $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$, so that becomes $\Gamma^2_{11} = \frac{1}{2}g^{2d}(\partial_1 g_{d1} + \partial_1 g_{d1} - \partial_d g_{11}) = \frac{1}{2}g^{21}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) + \frac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11})$. Discuss what I did with the d index and why here.
6. Notice from #4 that $g_{ij} = \begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix}$. Here, g^{ij} is the inverse matrix. What is it? Discuss and then respond on `pollev.com/drsarah314`:
   a) $\frac{1}{v^4}\begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix}$
   b) $v^4\begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix}$
   c) other
7. Here, the first coordinate for the Poincaré upper-half plane surface is u and the second coordinate is v, so rewriting the partials, can you see that we have
   $$\begin{aligned}\Gamma^2_{11} &= \tfrac{1}{2}g^{21}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) + \tfrac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) \\ &= \tfrac{1}{2}g^{21}(\tfrac{\partial}{\partial u}g_{11} + \tfrac{\partial}{\partial u}g_{11} - \tfrac{\partial}{\partial u}g_{11}) + \tfrac{1}{2}g^{22}(\tfrac{\partial}{\partial u}g_{21} + \tfrac{\partial}{\partial u}g_{21} - \tfrac{\partial}{\partial v}g_{11})?\end{aligned}$$

8. So that you don't have to flip back and forth, for the Poincaré upper-half plane surface $g_{ij} = \begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix}$ and $g^{ij} = \begin{bmatrix} v^2 & 0 \\ 0 & v^2 \end{bmatrix}$ and we are computing $\Gamma^2_{11} = \frac{1}{2}g^{21}(\frac{\partial}{\partial u}g_{11} + \frac{\partial}{\partial u}g_{11} - \frac{\partial}{\partial u}g_{11}) + \frac{1}{2}g^{22}(\frac{\partial}{\partial u}g_{21} + \frac{\partial}{\partial u}g_{21} - \frac{\partial}{\partial v}g_{11})$, so we'll substitute in for the inverse matrix g^{ij} first—the row and column numbers work just the same as they did in linear algebra:
   $\Gamma^2_{11} = \frac{1}{2}0(\frac{\partial}{\partial u}g_{11} + \frac{\partial}{\partial u}g_{11} - \frac{\partial}{\partial u}g_{11}) + \frac{1}{2}v^2(\frac{\partial}{\partial u}g_{21} + \frac{\partial}{\partial u}g_{21} - \frac{\partial}{\partial v}g_{11})$
   Since the first term in the sum is multiplied by 0, we don't need to compute anything further for that part and now we'll subsitute in the g_{ij} from the metric form matrix:
   $\Gamma^2_{11} = \frac{1}{2}v^2(\frac{\partial}{\partial u}g_{21} + \frac{\partial}{\partial u}g_{21} - \frac{\partial}{\partial v}g_{11}) = \frac{1}{2}v^2(\frac{\partial}{\partial u}0 + \frac{\partial}{\partial u}0 - \frac{\partial}{\partial v}\frac{1}{v^2})$
   There is only one nonzero partial to compute, so do that and reduce. What is Γ^2_{11}? Discuss and respond on `pollev.com/drsarah314`
   a) 0
   b) $\frac{1}{2}$
   c) $\frac{1}{v}$
   d) other

9. For the Poincaré upper-half plane surface, Maple computes Γ^2_{11} as above as well as
   $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0$ and
   $\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{v}$
   Geodesic equations as well as numerous curvatures are defined from the Christoffel symbols and their partial derivatives, building on one another:
   Riemann curvature tensor or Riemann-Christoffel tensor $R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$
   Ricci tensor $R_{ab} = R^c_{acb}$
   Scalar curvature $R = g^{ab}R_{ab}$
   Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$
   For the Riemann curvature tensor, let's look at one of the 16 possibilities: R^1_{212}. Notice $a = 1, b = 2, c = 1, d = 2$ in $R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$ and we'll sum over e in the last terms since it is in both their upper and lower indices. Therefore,
   $R^1_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{21} + \Gamma^e_{22}\Gamma^1_{e1} - \Gamma^e_{21}\Gamma^1_{e2} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{21} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22}$
   Next we substitute in the Christoffel symbols as well as the coordinates u for the 1st partial and v for the 2nd: $R^1_{212} = \frac{\partial}{\partial u}0 - \frac{\partial}{\partial v}(-\frac{1}{v}) + 00 + (-\frac{1}{v})(-\frac{1}{v}) - (-\frac{1}{v})(-\frac{1}{v}) - 00$. What does this reduce to?

10. Most of the 16 Riemann curvature tensor pieces vanish for the Poincaré upper-half plane surface, with the exception of R^1_{212} in #9, $R^1_{221} = R^2_{112} = \frac{1}{v^2}$ and $R^2_{121} = -\frac{1}{v^2}$.
    For the Ricci tensor, $R_{ab} = R^c_{acb}$ so $R_{11} = R^c_{1c1} = R^1_{111} + R^2_{121} = 0 + R^2_{121} = -\frac{1}{v^2}$.
    Compute $R_{22} = R^c_{2c2} = R^1_{212} + R^2_{222} = R^1_{212} + 0$ using your last response.

11. Next, let's look at the scalar curvature $g^{ab}R_{ab}$. Notice that a and b are both in the upper and lower indices, so we sum over them to obtain 4 terms: $g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22}$. There would be more for spacetime! Substitute the g^{ij} entries here as well as the Ricci curvature tensor components that connect to nonzero metric form entries.

12. Check that the scalar curvature is double the Gaussian curvature from #4.

13. Revise steps #5, 7 and 8 to compute $\Gamma^1_{12} = -\frac{1}{v}$ by hand and show work. Compare.

14. Show $R^2_{121} = -\frac{1}{v^2}$ by hand and show work.