Gauss-Bonnet on a Round Donut

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. Given the parametrization of the round torus as $x(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$ where $R > r > 0$ and are both constant, as a review of various definitions and computations in the surfaces module, look through the following computations related to the first fundamental form, discuss with your group, and write down any questions you have:

   $\vec{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$

   $\vec{x}_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$

   $E = \vec{x}_u \cdot \vec{x}_u = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u = r^2\sin^2 u(\cos^2 v + \sin^2 v) + r^2\cos^2 u$

   $F = \vec{x}_u \cdot \vec{x}_v = r\sin u\cos v(R + r\cos u)\sin v - r\sin u\sin v(R + r\cos u)\cos v + 0$

   $G = \vec{x}_v \cdot \vec{x}_v = (R + r\cos u)^2\sin^2 v + (R + r\cos u)^2\cos^2 v + 0$

   In addition, what do E, F, and G reduce to and (after this reduction) what does F tell us? Compare with your group as you discuss.

2. For the second fundamental form, we'll use the second partials dotted with U:

   $\vec{x}_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$

   $\vec{x}_{uv} = \vec{x}_{vu} = (r\sin u\sin v, -r\sin u\cos v, 0)$

   $\vec{x}_{vv} = (-(R + r\cos u)\cos v, -(R + r\cos u)\sin v, 0)$

   For U, first compute $\vec{x}_u \times \vec{x}_v$: $\begin{vmatrix} i & j & k \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$

   $= i(0 - r\cos u(R + r\cos u)\cos v)$
   $- j(0 + r\cos u(R + r\cos u)\sin v)$
   $+ k(-r\sin u\cos v(R + r\cos u)\cos v - r\sin u\sin v(R + r\cos u)\sin v)$
   $= (-r(R + r\cos u)\cos u\cos v, -r(R + r\cos u)\cos u\sin v, -r(R + r\cos u)\sin u(\cos^2 v + \sin^2 v))$
   $= (-r(R + r\cos u)\cos u\cos v, -r(R + r\cos u)\cos u\sin v, -r(R + r\cos u)\sin u)$

   Also, $|\vec{x}_u \times \vec{x}_v| = r(R + r\cos u)$ and so $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (-\cos u\cos v, -\cos u\sin v, -\sin u)$

   $l = \vec{x}_{uu} \cdot U = r\cos^2 u\cos^2 v + r\cos^2 u\sin^2 v + r\sin^2 u$

   $m = \vec{x}_{uv} \cdot U = r\sin u\sin v(-\cos u\cos v) - r\sin u\cos v(-\cos u\sin v) + 0$

   $n = \vec{x}_{vv} \cdot U = (R + r\cos u)\cos u\cos^2 v + (R + r\cos u)\cos u\sin^2 v + 0$

   What do l, m, and n reduce to? Compare with your group as you discuss.

3. Write out and reduce $K = \dfrac{ln - m^2}{EG - F^2}$

4. Next, use your prior response and discuss:

   - What happens to the sign of K when $-\frac{\pi}{2} < u < \frac{\pi}{2}$?
   - What happens to K when $u = \pm\frac{\pi}{2}$?
   - What happens to the sign of K when $\frac{\pi}{2} < u < \frac{3\pi}{2}$

5. Based on your last response as well as intuition about the principal normal curvatures on the physical donut—recall that $K = \kappa_1\kappa_2$ too, discuss and identify where on the physical donut the following are located:

   - $-\frac{\pi}{2} < u < \frac{\pi}{2}$
   - $u = \pm\frac{\pi}{2}$
   - $\frac{\pi}{2} < u < \frac{3\pi}{2}$

6. Using your computation for $K = \dfrac{ln - m^2}{EG - F^2}$, analyze what is the largest and smallest values of K analytically?

7. Next, identify where these are on the physical donut. Compare with your group as you discuss.

8. For the differential geometry side of Gauss-Bonnet, we'll look at $\int\int K\,dA$. The square root of the determinant of the first fundamental form gives us surface area and we can use that for dA on an infinitesimal level too: $\sqrt{EG - F^2}\,du\,dv$, so what is dA here?

9. Write out and simplify $\displaystyle\int_0^{2\pi}\int_0^{2\pi} K\,dA$.

10. Next, integrate and compare with your group.

11. Looking at the physical donut again, why does it make sense intuitively that the total Gaussian curvature is what you obtained with the integration?

12. The Gauss-Bonnet theorem in this setting lets us convert between differential geometry and topology via: $\int\int K dA = 2\pi\chi$, so what is χ for the round torus?

13. Can your group find a triangulation of the torus that demonstrates this Euler characteristic—where $\chi = V$ertices - Edges + Faces?

14. The Euler characteristic of a double torus is -2, so what does that tell you about K on this surface?

15. How about a surface with more holes, where the Euler characteristic is $2 - 2 \times$ number of holes?