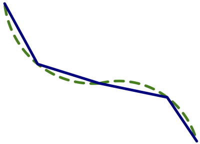
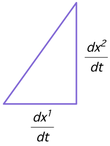
Arc Length, Tangent, and Physical Attributes

- arc length s
- unit tangent T
- velocity
- acceleration
- speed
- jerk

$$\text{arc length } s(t) = \int_a^t |\alpha'(u)| du$$

- measures the length of a curve by adding up infinitesimal linear approximation (Pythagorean theorem metric)

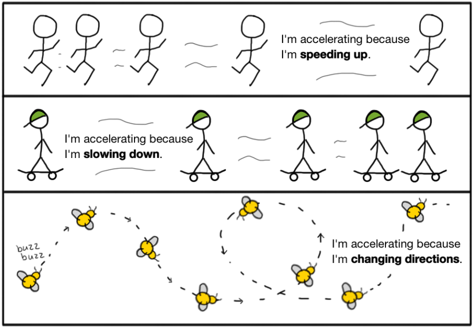




$$|\alpha'(t)| = \sqrt{(\tfrac{dx^1}{dt})^2 + (\tfrac{dx^2}{dt})^2 + \cdots}$$

Arc Length *s* Simplifies Computations

If α is a differentiable curve that is regular then α can be reparameterized by arc length *s* to have unit speed/tangent



https://www.khanacademy.org/science/physics/one-dimensional-motion/acceleration-tutorial/a/acceleration-article

Reparameterizing the Tractrix by Arc Length

- Compute arc length $s(t) = \int_{\frac{\pi}{2}}^{t} |\alpha'(\theta)| d\theta$
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- Reparameterize the curve by arc length $\beta(s) = \alpha(t(s))$

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$\rightarrow \beta(s) = (\cos(\arcsin e^{-s}) + \ln(\tan(\frac{\arcsin e^{-s}}{2})), \sin(\arcsin e^{-s}))$

Prove: If α is a differentiable curve that is regular then α can be reparameterized by arc length $s(t) = \int_a^t |\alpha'(u)|d$ to have unit speed/tangent.

Notice $s'(t) = \frac{ds}{dt} = |\alpha'(t)| > 0$ by def s, FTC, and regularity. To show s is strictly increasing, assume for contradiction $\exists x < y$ s.t. $s(x) \geq s(y)$.

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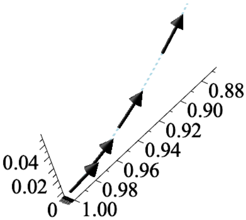
Typically we won't write down a closed form solution explicitly.

Frenet Frame: T in TNB

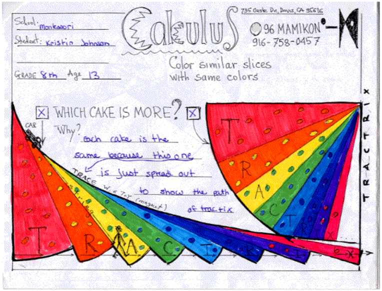
The unit tangent in the direction of motion is given a special name in differential geometry and its applications: T

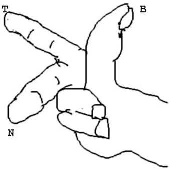
$T(s) = \alpha'(s)$

$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$ $\frac{velocity}{speed}$ as we can think of $\frac{\alpha'(t)}{\frac{ds}{dt}}$ as $\frac{d\alpha}{dt}\frac{dt}{ds}$



Differential Geometry and Physics of Tractrix





https://www.its.caltech.edu/~mamikon/Mont.html

http://www.rudyrucker.com/transrealbooks/collectedessays/images/kaptauhand.jpg

- velocity, acceleration, jerk, and higher time derivatives
- speed and arc length
- TNB Frame, curvature and torsion