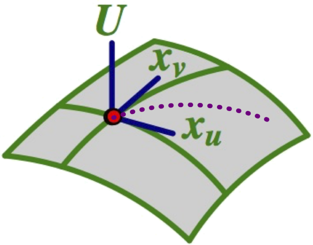
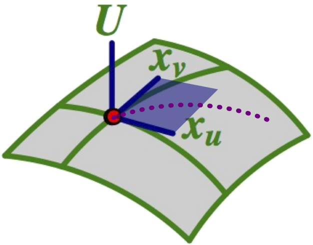
First Fundamental Form





- Regular surface $M = \mathbf{x}(u, v)$, where $\vec{x}_u \times \vec{x}_v \neq 0$, and $u(t)$ & $v(t)$ give curve $\alpha(t)$. Then $\vec{x}_u, \vec{x}_v$ form basis for T_pM and $\alpha'(t) =$

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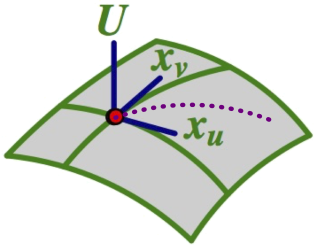

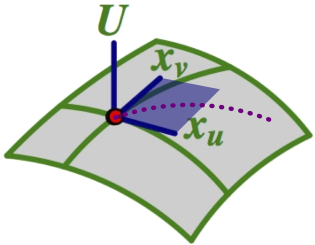

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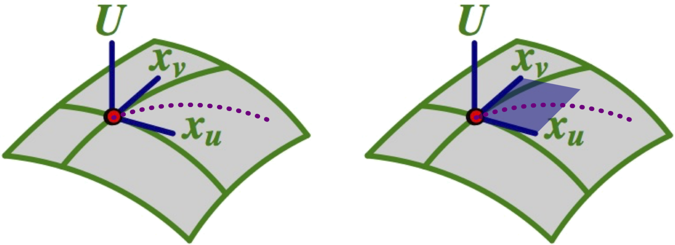

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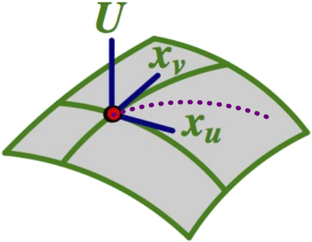

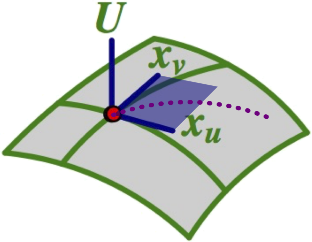

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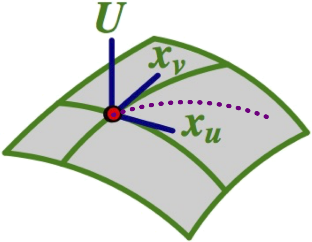

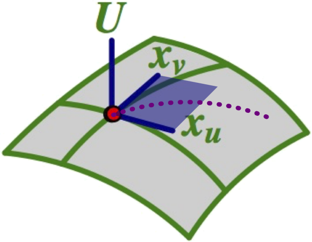

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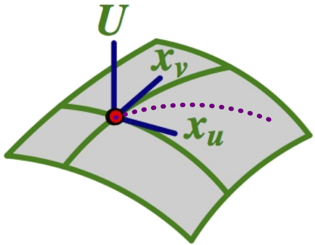
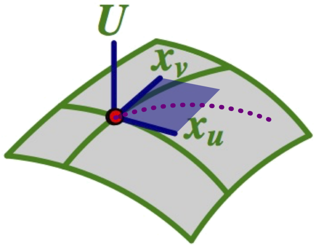
  $= \vec{x_u} \cdot \vec{x_u}(\frac{du}{dt})^2 + 2\vec{x_u} \cdot \vec{x_v}\frac{du}{dt}\frac{dv}{dt} + \vec{x_v} \cdot \vec{x_v}(\frac{dv}{dt})^2$

First Fundamental Form





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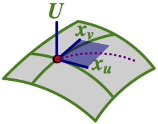
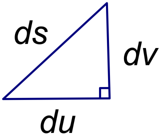
$= E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$
$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^idu^j$

First Fundamental Form for Plane and Cone

$\mathbf{x}(u, v) = (u, v, 0)$ compared to $\mathbf{x}(u, v) = (u \cos v, u \sin v, u)$





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$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^i du^j$

plane $x_u = (1, 0, 0)$ and $x_v = (0, 1, 0)$

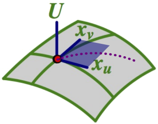
$x_u \cdot x_u = E = g_{11} = 1$

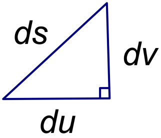
$x_u \cdot x_v = F = g_{12} = g_{21} = 0$

$x_v \cdot x_v = G = g_{22} = 1$ so $ds^2 = du^2 + dv^2$

First Fundamental Form for Plane and Cone

$\mathbf{x}(u,v) = (u,v,0)$ compared to $\mathbf{x}(u,v) = (u\cos v, u\sin v, u)$





$(\frac{ds}{dt})^2 = E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$

$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^idu^j$

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$x_v \cdot x_v = G = g_{22} = 1$ so $ds^2 = du^2 + dv^2$

cone $x_u = (\cos v, \sin v, 1)$ and $x_v = (-u\sin v, u\cos v, 0)$

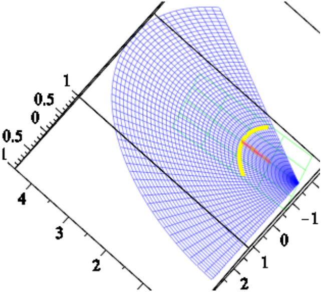
$x_u \cdot x_u = E = g_{11} = 2$

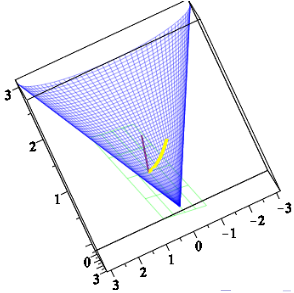
$x_u \cdot x_v = F = g_{12} = g_{21} = 0$

$x_v \cdot x_v = G = g_{22} = u^2$ so $ds^2 = 2du^2 + u^2dv^2$

New Plane Isometric to Cone

- new plane $[\sqrt{2}u\cos(\frac{v}{\sqrt{2}}), \sqrt{2}u\sin(\frac{v}{\sqrt{2}}), 0]$ that is isometric to the cone $[u\cos(v), u\sin(v), u]$
- longitude and latitude on the new plane
- first fundamental form of the new plane and cone
- using secant to write the geodesic between the points $(1,0,1)$ and $(0,1,1)$ on the cone (i.e. the point $x = 1$ and $y = 0$ and the point $x = 1, y = \frac{\pi}{2}$





First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$

- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
- g_{ij} determines dot products of tangent vectors $\vec{w}_1$, $\vec{w}_2$ in T_pM

$\{\vec{x_u}, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$, $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

$\vec{w}_1 \cdot \vec{w}_2 \overset{\text{foil}}{=}$

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$$\vec{w}_1 \cdot \vec{w}_2 \overset{\text{foil}}{=} ac\vec{x}_u \cdot \vec{x}_u + (ad + bc)\vec{x_u} \cdot \vec{x}_v + bd\vec{x}_v \cdot \vec{x}_v$$
$$= acE + (ad + bc)F + bdG$$

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&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} cE + dF \\ cF + dG \end{bmatrix} \\
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\end{aligned}$$

$= \vec{w}_1 \cdot \vec{w}_2 = |\vec{w}_1||\vec{w}_2| \cos\theta$

E, F, G play important roles in many intrinsic properties of a surface like length $(\frac{ds}{dt})^2$, area (det) and angles (above)

Geodesic Curvature Depends only on the Metric

$\vec{\kappa}_\alpha$ (curve's curvature vector): $\frac{T'(t)}{|\alpha'(t)|}$
$\vec{\kappa}_n$ (normal curvature): projection of $\vec{\kappa}_\alpha$ onto $U = (U \cdot \vec{\kappa}_\alpha)U$
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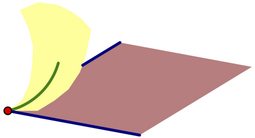
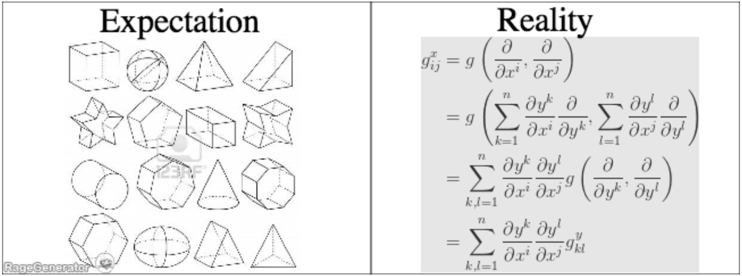


$\alpha'(t) \overset{\text{chain rule}}{=} \vec{x}_u \frac{du}{dt} + \vec{x}_v \frac{dv}{dt} = \vec{x}_u u' + \vec{x}_v v'$ or equivalently
$\dot{\alpha} = \vec{x}_u \dot{u} + \vec{x}_v \dot{v}$

$\vec{\kappa}_g$ when $F = 0$:
$\sqrt{EG}(-\frac{E_v}{2G}u'^3 + (\frac{G_u}{G} - \frac{E_u}{2E})u'^2 v' + (\frac{G_v}{2G} - \frac{E_v}{E})u' v'^2 + \frac{G_u}{2E}v'^3 +$
$u' v'' - u'' v')$



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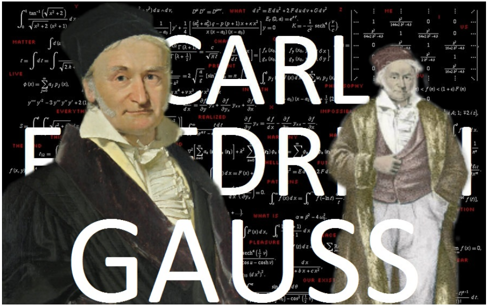

http://vignette1.wikia.nocookie.net/epicrapbattlesofhistory/images/8/83/Pizap.

com14239077419801.jpg

Spherical First Fundamental Form

geographical coordinates

$\mathbf{x}(u, v) = (r\cos u \cos v, r\sin u \cos v, r\sin v)$

$\vec{x}_u = (-r\sin u \cos v, r\cos u \cos v, 0)$

$\vec{x}_v = (-r\cos u \sin v, -r\sin u \sin v, r\cos v)$

- What is $E = \vec{x}_u \cdot \vec{x}_u$, $F = \vec{x}_u \cdot \vec{x}_v$, and $G = \vec{x}_v \cdot \vec{x}_v$?
- Interpret F—what does it tell us about the relationship between $\vec{x}_u$ and $\vec{x}_v$?

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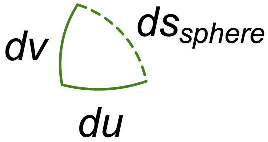
- Interpret F—what does it tell us about the relationship between $\vec{x}_u$ and $\vec{x}_v$?
- $\begin{bmatrix} r^2\cos^2 v & 0 \\ 0 & r^2 \end{bmatrix}$

  $ds^2 = r^2\cos^2 v du^2 + r^2 dv^2$

  When $r = 1$ then $ds^2 = \cos^2 v du^2 + dv^2$



Implications of the Spherical Metric Form

$(\frac{ds}{dt})^2 = E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$

$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1 du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij} du^i du^j$

$ds^2_{\text{sphere}} = \cos^2 v du^2 + dv^2 < du^2 + dv^2 = ds^2_{\text{plane}}$

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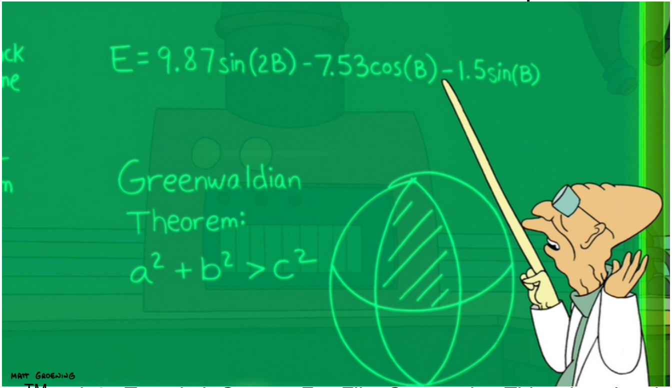
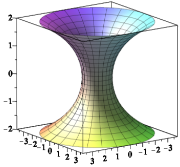
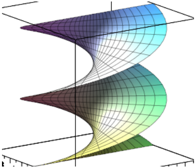


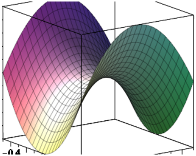
Implications of the Metric Form



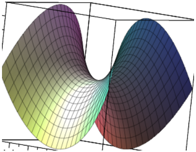
catenoid



helicoid



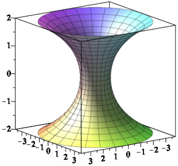
saddle

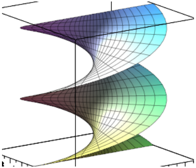


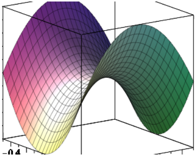
Enneper's surface

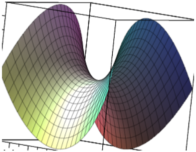
catenoid $\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$
helicoid $\mathbf{x}(u, v) = (\sinh u \cos v, \sinh u \sin v, v)$
saddle $\mathbf{x}(u, v) = (u, v, u^2 - v^2)$
Enneper's surface $\mathbf{x}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$

Implications of the Metric Form


catenoid


helicoid


saddle


Enneper's surface

catenoid $\mathbf{x}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$
helicoid $\mathbf{x}(u,v) = (\sinh u \cos v, \sinh u \sin v, v)$
saddle $\mathbf{x}(u,v) = (u, v, u^2 - v^2)$
Enneper's surface $\mathbf{x}(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$

`http://virtualmathmuseum.org/Surface/helicoid-catenoid/helicoid-catenoid.mov`