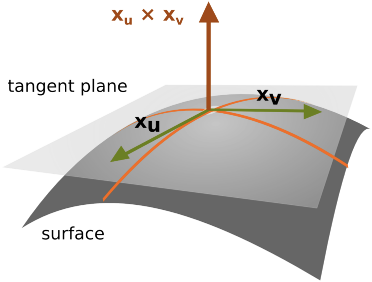
Coordinates and the Unit Normal U



https://www.cse.iitb.ac.in/~cs749/spr2017/handouts/vas_surf1.pdf

coordinates for surface in $\mathbb{R}^3$

$(u, v) \to x(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$

- coordinate curves $x(u, \text{constant})$ and $x(\text{constant}, v)$
- $\vec{x}_u$ and $\vec{x}_v$ are tangent vectors to the coordinate curves
- T_pM is $((x, y, z) - \vec{p}) \cdot x_u \times x_v = 0$
- The *unit normal* to the surface at a point is $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$ determines the tangent plane

Extrinsic $\mathbb{R}^3$ Coordinates on a Cylinder

$x(\theta, z) = (rcos(\theta), rsin(\theta), z)$, r constant

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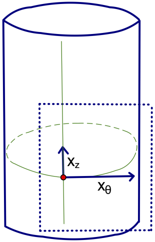
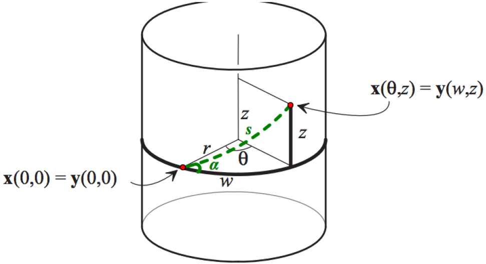


- T_pM is $((x, y, z) - \vec{p}) \cdot U = 0$ and $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$

$$\vec{x}_u \times \vec{x}_v = \begin{vmatrix} i & j & k \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

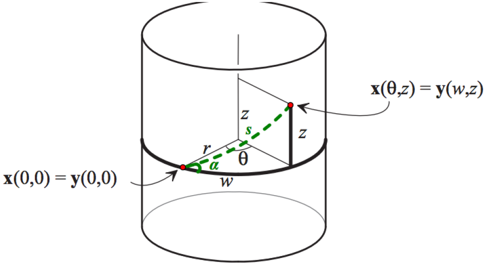
Intrinsic Coordinates and Geodesic Equation



Adapted http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf

- Choose (0,0) as an intrinsic origin.

Intrinsic Coordinates and Geodesic Equation



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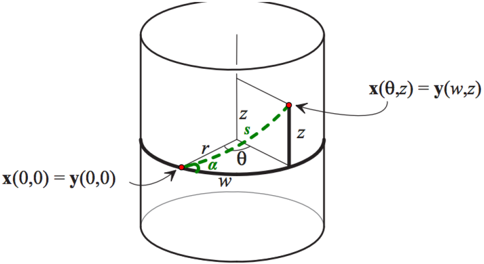
- Choose (0,0) as an intrinsic origin. There is 1 geodesic that will return there, so call that the base curve
- Choose +z as a direction $\perp$ to the base curve

Intrinsic Coordinates and Geodesic Equation



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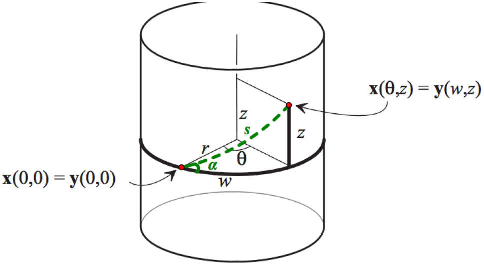
- Choose (0,0) as an intrinsic origin. There is 1 geodesic that will return there, so call that the base curve
- Choose +z as a direction ⊥ to the base curve
- Parameterize geodesic $\gamma(s) = (s\cos\alpha, s\sin\alpha)$.

Intrinsic Coordinates and Geodesic Equation



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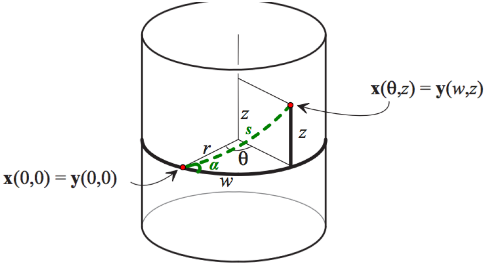
Intrinsic Coordinates and Geodesic Equation



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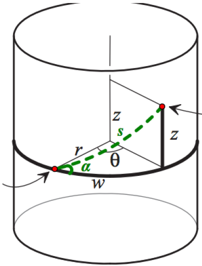

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- Parameterize geodesic $\gamma(s) = (s\cos\alpha, s\sin\alpha)$. Then $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{s\sin\alpha}{s\cos\alpha} = \frac{z}{w}$
- Back in extrinsic coordinates, $r\theta = w, z = z$ so $\theta = \frac{w}{r} = \frac{s\cos\alpha}{r}$ and $\gamma(s) = (r\cos\frac{s\cos\alpha}{r}, r\sin\frac{s\cos\alpha}{r}, s\sin\alpha)$

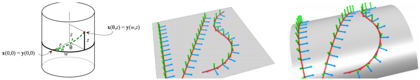
Coordinates and Geodesic Curvature

coordinates for surface in $\mathbb{R}^3$

$(u, v) \to x(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$

- coordinate curves $x(u, \text{constant})$ and $x(\text{constant}, v)$
- $\vec{x}_u$ and $\vec{x}_v$ are tangent vectors to the coordinate curves
- for regular surfaces in $\mathbb{R}^3$, T_pM is $((x, y, z) - \vec{p}) \cdot U = 0$
- The *unit normal* to the surface at a point is $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$ determines the tangent plane
- If $\vec{\kappa}_\alpha$ is the curvature vector for a curve $\alpha(t)$ on the surface then the *normal curvature* is the projection onto U:
$$\vec{\kappa}_n = (U \cdot \vec{\kappa}_\alpha)U$$
- The *geodesic curvature* is what is felt by the bug (in the tangent plane T_pM): $$\vec{\kappa}_g = \vec{\kappa}_\alpha - \vec{\kappa}_n$$

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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$\vec{x}_u = (-sin(u), cos(u), 0), \vec{x}_v = (0, 0, 1)$.

$U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (cos(u), sin(u), 0)$

$\vec{\kappa}_\alpha$ (curve's curvature vector): $\frac{T'(t)}{|\alpha'(t)|}$

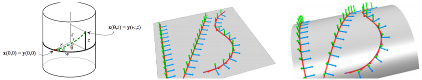
$\vec{\kappa}_n$ (normal curvature): projection of $\vec{\kappa}_\alpha$ onto $U = (U \cdot \vec{\kappa}_\alpha)U$

$\vec{\kappa}_g$ (geodesic curvature): $\vec{\kappa}_\alpha$ - $\vec{\kappa}_n$

Ex: $\gamma(t) = (\cos t, \sin t, t)$

Calculate $\vec{\kappa}_\gamma = \frac{T'(t)}{|\gamma'(t)|}$ and compare with U to explain why it isn't felt by the bug

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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Ex: $\gamma(t) = (\cos t, \sin t, t)$

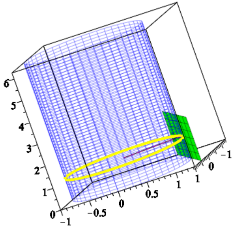
Calculate $\vec{\kappa}_\gamma = \frac{T'(t)}{|\gamma'(t)|}$ and compare with U to explain why it isn't felt by the bug

$\gamma'(t) = (-\sin t, \cos t, 1)$ and $|\gamma'(t)| = \sqrt{2}$, so

$T = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$ and $\vec{\kappa}_\gamma = \frac{1}{2}(-\cos t, -\sin t, 0)$

Maple File on Geodesic and Normal Curvatures

adapted from David Henderson



$\vec{\kappa}_\alpha$ pink dashed thickness 1
$\vec{\kappa}_n$ black solid thickness 2
$\vec{\kappa}_g$ tan dashdot style thickness 4

- The *unit normal* to the surface at a point is $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$
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Commands for Maple File on Curvatures

```
g := (x,y) -> [cos(x), sin(x), y]:
a1:=0: a2:=2*Pi: b1:=0: b2:=4:
c1 := 1: c2 := 3:
Point := 2:
f1:= (t) -> t:
f2:= (t) -> 1:
```

```
g := (x,y) -> [cos(x), sin(x), y]:
a1:=0: a2:=2*Pi: b1:=0: b2:=4:
c1 := 1: c2 := 3:
Point := 2:
f1:= (t) -> t:
f2:= (t) -> sin(t):
```

Commands for Maple File on Curvatures

```
g := (x,y) -> [cos(x), sin(x), y]:
a1:=0: a2:=2*Pi:  b1:=0: b2:=4:
c1 := 1: c2 := 20:
Point := 2:
f1:= (t) -> t*cos(Pi/16):
f2:= (t) -> t*sin(Pi/16):
```

Commands for Maple File on Curvatures

```
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a1:=0: a2:=2*Pi:  b1:=0: b2:=4:
c1 := 1: c2 := 20:
Point := 2:
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