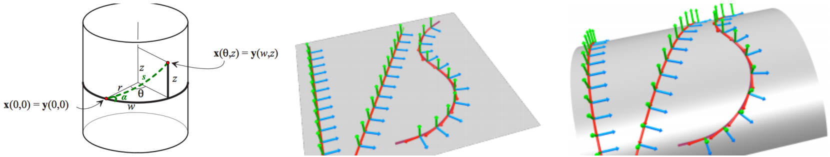
The Speed *v* of a Geodesic



Adapted `http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf`

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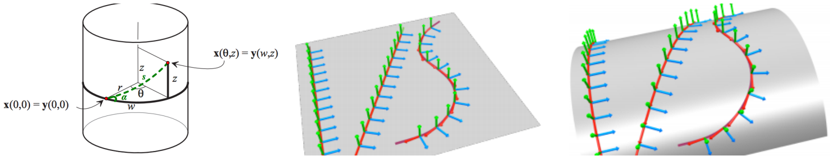

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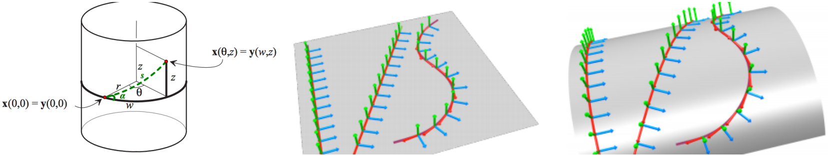
$v = |\alpha'(t)| = |\vec{v}|, \qquad T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{v(t)}$ so $\alpha'(t) = v(t)T(t)$

$\alpha''(t)$

The Speed v of a Geodesic



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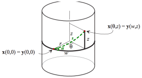
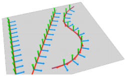
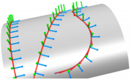
$\alpha''(t) = v'(t)T(t) + v(t)T'(t)$

$v'(t)$: linear or tangential acceleration (tangential component of acceleration vector)

For a geodesic, since we don't feel any curvature in the tangent plane—only normal to the surface— $v'(t) = 0$ so v is constant.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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$x(u, v) = (cos(u), sin(u), v)$ Normal U to the surface?

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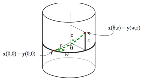

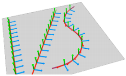
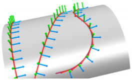

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Ex 1: $\alpha(t) = (cos(t), sin(t), sin(t))$.

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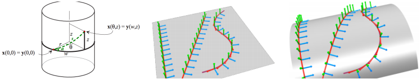

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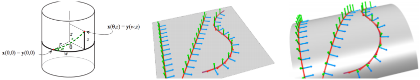

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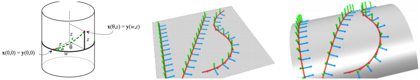

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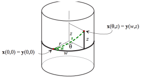

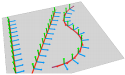
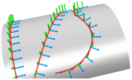

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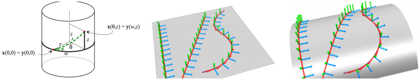
Ex 2: $\gamma(t) = (cos(t), sin(t), t)$ Calculate $\vec{\kappa} = \frac{T'(t)}{|\gamma'(t)|}$ and compare with U to explain why it isn't felt by the bug

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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$\vec{\kappa}_n$ (normal curvature): projection of $\vec{\kappa}_\alpha$ onto U = $(U \cdot \vec{\kappa}_\alpha)U$

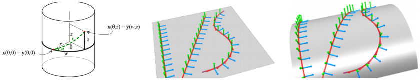
$\vec{\kappa}_g$ (geodesic curvature): $\vec{\kappa}_\alpha$ - $\vec{\kappa}_n$

Ex 3: $\gamma(t) = (cos(t), sin(t), 0)$ is a geodesic.

$\frac{\gamma'(t)}{|\gamma'(t)|} = T = (-sin(t), cos(t), 0)$ (speed is 1).

$\vec{\kappa} = \frac{T'(t)}{|\gamma'(t)|} = (-cos(t), -sin(t), 0)$ no T_pM component, only U

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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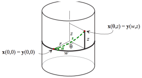

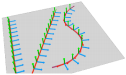
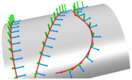

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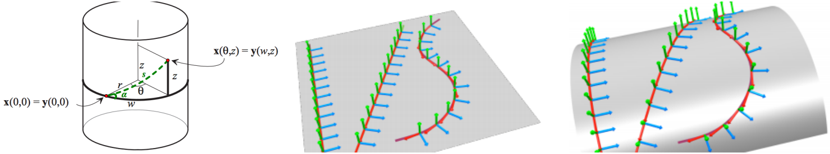
Classifying Cylinder Geodesics Using α''



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surface $x(u,v) = (cos(u), sin(u), v)$—two free variables u, v

$\vec{x}_u = (-sin(u), cos(u), 0), \vec{x}_v = (0, 0, 1).$

$U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (cos(u), sin(u), 0)$

curve on surface $\alpha(t) = (cos(u(t)), sin(u(t)), v(t))$

$\alpha'(t) = (-\sin u \;\; u', \cos u \;\; u', v'),$

$\alpha''(t) = (-\sin u \;\; u'' - \cos u \;\; u'u', \cos u \;\; u'' - \sin u \;\; u'u', v'')$

$\alpha''(t)_{\text{tangential}} = (-\sin u \;\; u'', \cos u \;\; u'', v'')$

so $u'' = 0$ and $v'' = 0$ and $u = at + a_0, v = bt + b_0$

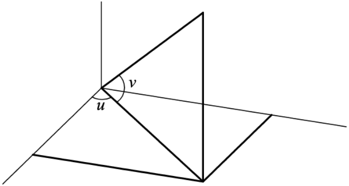
$\gamma(t) = (\cos(at + a_0), \sin(at + a_0), bt + b_0)$

Spherical Coordinates

geographical coordinates

$\mathbf{x}(u, v) = (r\cos u\cos v, r\sin u\cos v, r\sin v)$

- role of coordinates: hold one constant and explain what kind of curve the other gives, and then the reverse.



Differential Geometry and Its Applications by John Oprea

- What is U?

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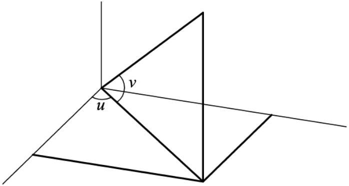

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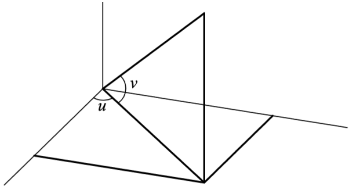

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  $x_u \times x_v = (r^2 \cos u \cos^2 v, r^2 \sin u \cos^2 v, r^2 \cos v \sin v)$

  $= r^2 \cos v(\cos u \cos v, \sin u \cos v, \sin v)$

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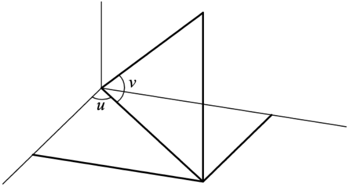

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  $= r^2 \cos v(\cos u \cos v, \sin u \cos v, \sin v)$
  $|x_u \times x_v| = r^2 \cos v$ so $U = (\cos u \cos v, \sin u \cos v, \sin v)$

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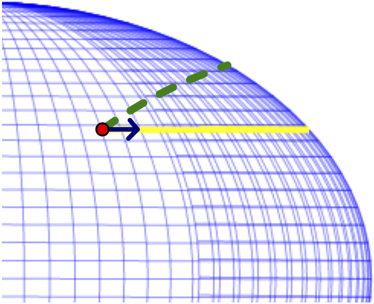

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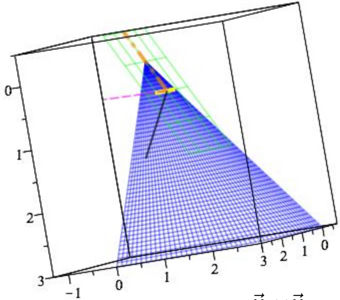
Maple File on Geodesic and Normal Curvatures

adapted from David Henderson



$\vec{\kappa}_\alpha$ pink dashed thickness 1
$\vec{\kappa}_n$ black solid thickness 2
$\vec{\kappa}_g$ tan dashdot style thickness 4

- The *unit normal* to the surface at a point is $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$
- If $\vec{\kappa}_\alpha$ is the curvature vector for a curve $\alpha(t)$ on the surface then the *normal curvature* is the projection onto U:
$$\vec{\kappa}_n = (U \cdot \vec{\kappa}_\alpha)U$$
- The *geodesic curvature* is what is felt by the bug (in the tangent plane T_pM): $$\vec{\kappa}_g = \vec{\kappa}_\alpha - \vec{\kappa}_n$$

Geodesics on a Sphere are Great Circles

Let $\gamma(s)$ be a geodesic on the geographic sphere. We'll show it must be a great circle. It has constant speed, so we can reparameterize in s. From our computations, $U = \frac{\gamma}{r}$. Then

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$\gamma'' = (\gamma'' \cdot U)U =$

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where θ is the angle between γ'' and γ'. Taking the magnitude,
$|\gamma''| = \frac{1}{r^2}|\gamma''||\gamma|| \cos\theta||\gamma|$

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$\gamma'' = (\gamma'' \cdot U)U = (\gamma'' \cdot \frac{\gamma}{r})\frac{\gamma}{r} = \frac{1}{r^2}(\gamma'' \cdot \gamma)\gamma = \frac{1}{r^2}(|\gamma''||\gamma| \cos\theta)\gamma$,
where θ is the angle between γ'' and γ'. Taking the magnitude,
$|\gamma''| = \frac{1}{r^2}|\gamma''||\gamma|| \cos\theta||\gamma| = \frac{1}{r^2}|\gamma''|r| \cos\theta|r = |\gamma''|| \cos\theta|$ so
$| \cos\theta| = 1$ and γ'' and γ are parallel!
Moreover,

Geodesics on a Sphere are Great Circles

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$|\gamma''| = \frac{1}{r^2}|\gamma''||\gamma||\cos\theta||\gamma| = \frac{1}{r^2}|\gamma''|r|\cos\theta|r = |\gamma''||\cos\theta|$ so
$|\cos\theta| = 1$ and γ'' and γ are parallel!
Moreover, $\gamma'' = T' = \kappa N$ and $\gamma = rU$ since $U = \frac{\gamma}{r}$, so κN and rU are parallel. But N and U are both unit vectors so
$U' = \pm N' = \pm(-\kappa T + \tau B)$ and U' also equals $\frac{\gamma'}{r} = \frac{T}{r}$. But T and B are perpendicular so T can't have a B component. Thus $\tau = 0$ and $|\kappa| = \frac{1}{r}$. We previously proved this was part of a circle. The radius of the circle is the full r, i.e. a great circle on the sphere. $\square$