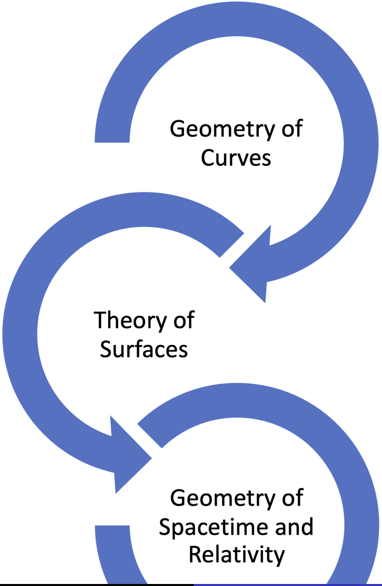
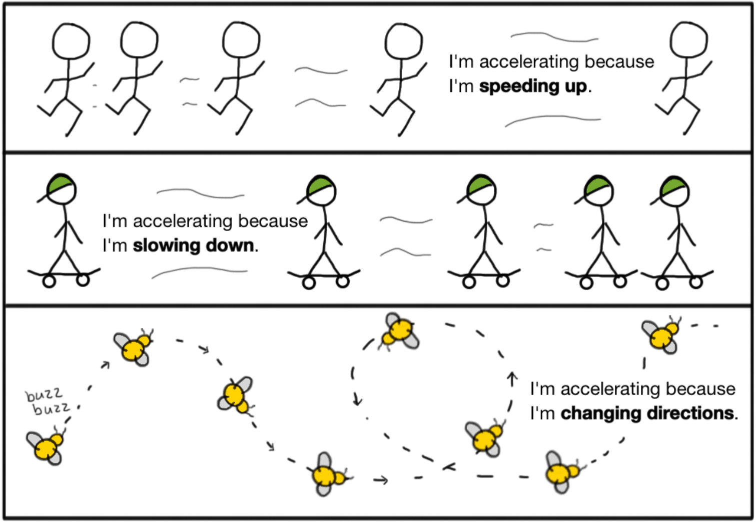
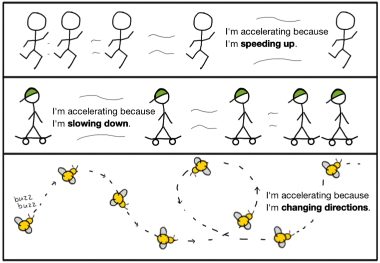
Differential Geometry of a Line



Prove that $\alpha(t)$ is a curve that is a constant speed straight line iff the acceleration is $\vec{0}$.



https://www.khanacademy.org/science/physics/one-dimensional-motion/acceleration-tutorial/a/acceleration-article



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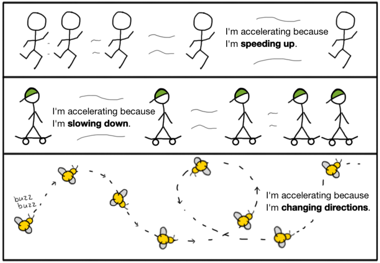
Prove that $\alpha(t)$ is a curve that is a constant speed straight line $\iff$ the acceleration is $\vec{0}$.

$\implies \alpha(t) = \vec{p} + t\vec{v}$



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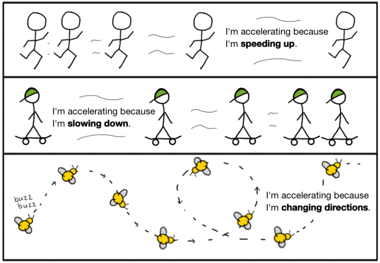
$$\implies \alpha(t) = \vec{p} + t\vec{v}$$
$$\alpha'(t) = \vec{0} + \vec{v}$$



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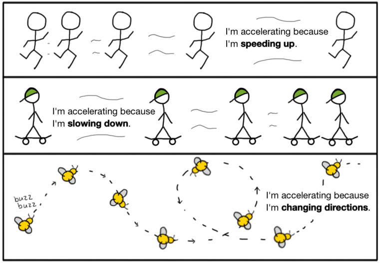
Prove that $\alpha(t)$ is a curve that is a constant speed straight line $\iff$ the acceleration is $\vec{0}$.

$$\Longrightarrow \quad \alpha(t) = \vec{p} + t\vec{v}$$

$$\alpha'(t) = \vec{0} + \vec{v}$$

$$\alpha''(t) = \vec{0}$$

$$\Longleftarrow \quad \alpha''(t) = \vec{0}$$



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Prove that $\alpha(t)$ is a curve that is a constant speed straight line $\iff$ the acceleration is $\vec{0}$.

$$\Longrightarrow \quad \begin{aligned} \alpha(t) &= \vec{p} + t\vec{v} \\ \alpha'(t) &= \vec{0} + \vec{v} \\ \alpha''(t) &= \vec{0} \end{aligned}$$

$$\Longleftarrow \quad \begin{aligned} \alpha''(t) &= \vec{0} \\ \alpha'(t) &= \int \alpha''(t)dt = \int \vec{0}dt = \vec{v} \\ \alpha(t) &= \int \alpha'(t)dt = \int \vec{v}dt = t\vec{v} + c \end{aligned}$$

Why is a $\vec{p} + t\vec{v}$ line the shortest distance path between 2 points in Euclidean geometry?

Intuition?

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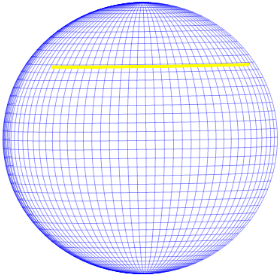
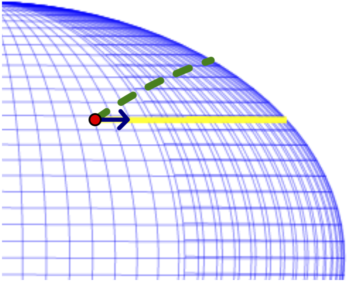




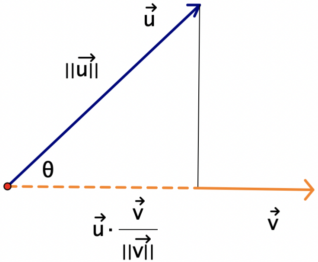
Intuition?

Why is a line $l(t) = \vec{p} + t(\vec{q} - \vec{p})$ shorter than any other curve $\alpha(t)$ between $\vec{p}$ and $\vec{q}$ in Euclidean geometry?

- dot product of two vectors in $\mathbb{R}^3$

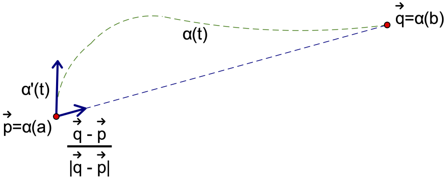
$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = u^1 v^1 + u^2 v^2 + u^3 v^3$$

- another formulation of the dot product $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them.



Why is a line $l(t) = \vec{p} + t(\vec{q} - \vec{p})$ shorter than any other curve $\alpha(t)$ between $\vec{p}$ and $\vec{q}$ in Euclidean geometry?

- dot product $\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos\theta = \sum v_1^i v_2^i$
- magnitude, norm, or length of a vector $|\vec{v}| = ||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\vec{v}^T \vec{v}}$
- normalize or unitize a vector, e.g. $\vec{q} - \vec{p} \to \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$
- derivative dot product of 2 curves in $\mathbb{R}^3$ $\alpha(t) \cdot \beta(t)$ $= \frac{d}{dt}(\alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3) = \frac{d}{dt}\sum_i \alpha^i\beta^i = \sum_i \frac{d}{dt}(\alpha^i\beta^i)$ $= \sum_i(\frac{d\alpha^i}{dt}\beta^i + \alpha^i\frac{d\beta^i}{dt}) = \sum_i \frac{d\alpha^i}{dt}\beta^i + \sum_i \alpha^i\frac{d\beta^i}{dt}$ $= \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t)$
- arc length $\int_a^b |\alpha'(t)|dt$ line length $|\vec{q} - \vec{p}|$



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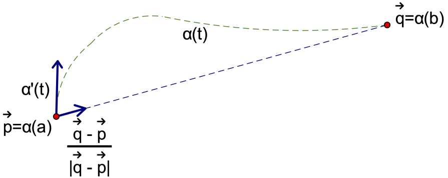
$|\vec{q} - \vec{p}| = \frac{|\vec{q}-\vec{p}|^2}{|\vec{q}-\vec{p}|}$



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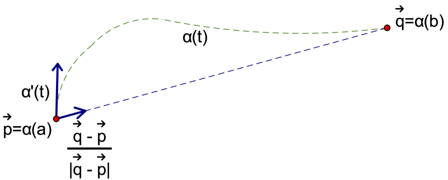

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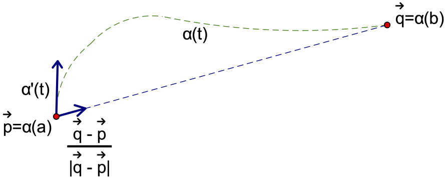
$= \vec{q} \cdot \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} - \vec{p} \cdot \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} = \alpha(b) \cdot \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} - \alpha(a) \cdot \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$



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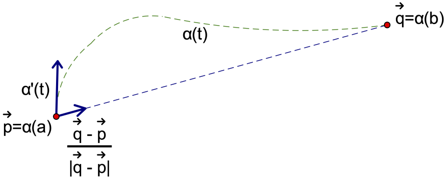

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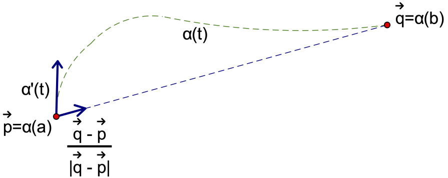

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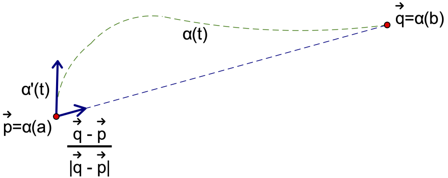
$$= \int_a^b |\alpha'(t)| |\frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}| \cos\theta(t) dt$$



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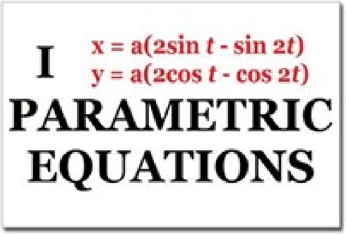
$$\leq \int_a^b |\alpha'(t)| |\frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}| dt = \int_a^b |\alpha'(t)| dt$$



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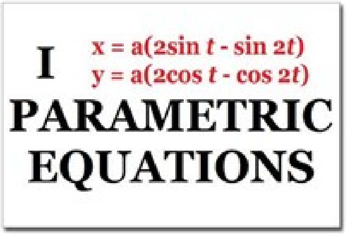
with(Student[VectorCalculus]):

```
TNBFrame(<2*sin(t)-sin(2*t),2*cos(t)-cos(2*t),0>,
range=0..3*Pi,output=animation,
scaling=constrained,axes=frame,frames=50);
```



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