Gaussian and Mean Curvature

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. It's π-day! A circle can be defined as the set of points equidistant from a center. On the physical sphere, use the string as the length of the radius to consider the set of points equidistant from the north pole. Discuss what do possible spherical circles look like?

2. Keeping to the surface of the sphere (no drilling through allowed!), consider the circumference of a spherical circle over its diameter (that's the definition of π—in flat spaces with the Euclidean metric that satisfy the Pythagorean theorem it is the familiar $3.14159...$, an irrational number). What happens to "π" on the sphere for spherical circles close to the North pole? How about for spherical circles close to the equator? Discuss.

3. Review as you fill in the blanks and discuss:
   - The first fundamental form can be represented by the matrix $\begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$ where $E = \vec{x}_u \cdot \vec{x}_u$, $F =$ __________ and $G = \vec{x}_v \cdot \vec{x}_v$
   - The shape operator can also be represented by a matrix of weights from $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$ and $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ when they are written as linear combinations in the basis vectors _____ and _____
   - The second fundamental form can be represented by the matrix $\begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$ where $l = S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U$, $m =$ ____________________ and $n = S(\vec{x}_v) \cdot \vec{x}_v = \vec{x}_{vv} \cdot U$.

   Which matrix representations are symmetric, with $A = A^T$, i.e. if we reflect the matrix about the main diagonal of the matrix, swapping every row with every column to obtain the transpose, when do we obtain the same matrix we started with? Discuss and respond on `pollev.com/drsarah314`

   a) first fundamental form

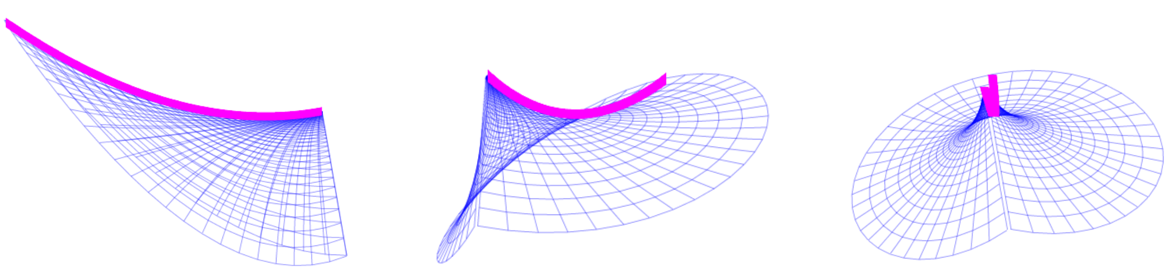
   b) shape operator

   c) second fundamental form

   d) all of the above

   e) other

4. For the Catalan surface, Maple outputs mean curvature 0 everywhere. The mean curvature H is the average of the principal curvatures, the maximum and minimum normal curvatures at a point, the eigenvalues of the shape operator, and 0 mean curvature surfaces locally minimize area. Here are three views of a parabola on the Catalan surface, along $u = \pi$ and $v = t$. Discuss intuition for the principal curvatures along points on this geodesic that is a parabola—are their signs the same or opposite if they are nonzero?



5. For the Catalan surface, Maple outputs a long expression for the Gaussian curvature K, the product of the principal curvatures:

$$\frac{(((8\cosh^3 v\cos u-4\cos^2 u-12\cosh^2 v+8)\cos^2\frac{u}{2}+4\sin\frac{u}{2}\sin u(-1+\cosh v)(\cosh v+1)(\cos u-\cosh v)\cos\frac{u}{2}+5\cosh^4 v-10\cosh^3 v\cos u+(\cos^2 u+3)\cosh^2 v+2\cos u\cosh v+3\cos^2 u-4)\cosh^2\frac{v}{2}+4\sinh\frac{v}{2}\sinh v((\cos u\cosh v-1)(\cos u-\cosh v)\cos^2\frac{u}{2}+(2\sin u\cosh^2 v-2\sin u)\sin\frac{u}{2}\cos\frac{u}{2}-\cosh v(\cos u-\cosh v)^2)\cosh\frac{v}{2}+(\cosh v+1)((\cos^2 u-2\cos u\cosh v-3\cosh^2 v+4)\cos^2\frac{u}{2}-(\cos u-\cosh v)^2)(-1+\cosh v))}{(16((-4cos^2\frac{u}{2}+4)\cosh^2\frac{v}{2}+\cosh^2 v-1)(((\cos u\cosh v-1)\cos^2\frac{u}{2}-\cos u\cosh v+\frac{\cosh^2 v}{2}+1/2)\cosh^2\frac{v}{2}+\sin\frac{u}{2}\cosh\frac{v}{2}\sin u\cos\frac{u}{2}\sinh\frac{v}{2}\sinh v+((-1+\cosh v)(\cosh v+1)(\cos u-\cosh v+2\cos\frac{u}{2})(\cos u-\cosh v-2\cos\frac{u}{2}))/8)(-2\cosh^2\frac{v}{2}\cos^2\frac{u}{2}+\cos u\cosh v-\frac{\cosh^2 v^2}{2}+2\cos^2\frac{u}{2}-\frac{1}{2}))}$$

According to Wolfram MathWorld, the Gaussian curvature can be reduced to $\dfrac{1}{\cosh^4\frac{v}{2}8(\cos u-\cosh v)}$.

The reference for the reduced version is

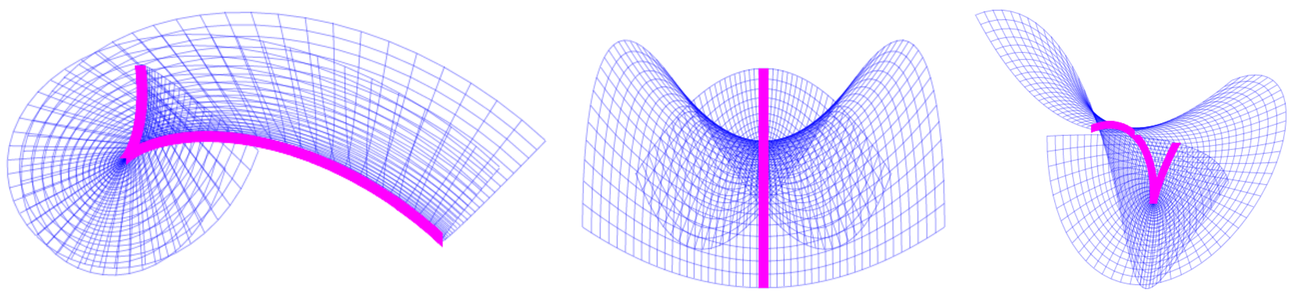
Weisstein, Eric W. "Catalan's Surface." From MathWorld—A Wolfram Web Resource.

`https://mathworld.wolfram.com/CatalansSurface.html`

Recall that $\cosh$ has a minimum of 1. So, for points where it is defined, and arguing analytically, what is the sign of the Gaussian curvature K?

6. Does the analytical reasoning here mesh with the geometric intuition from the earlier question?

7. Here are some images of $u = t$ and $v = 0$ on the Catalan surface, where we see a geodesic that is a cycloid!



Discuss intuition for the sign of the Gaussian curvature K on points along this curve.

8. For the physical sphere, discuss intuition for the principal curvatures.

9. Using this intuition, for a sphere of radius r what can we say about K and $|H|$? Discuss and then respond on `pollev.com/drsarah314` to let me know you have discussed this:

   a) we have discussed this

10. Discuss intuition for the principal curvatures on a cylinder of radius r. Using this intuition, what can we say about K and $|H|$?

11. Discuss intuition for the principal curvatures on a horse's saddle. Using this intuition, what can we say about the Gaussian curvature?

12. While the principal curvatures can change with the embedding, the Gauss curvature does not. Discuss how this can connect to folding thin but flexible crust pizza, which some people do as they are eating, like in:

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13. Review how we derived l.