Surfaces Not Embedded in $\mathbb{R}^3$ and Preparing for In-class Surfaces Assessment

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. The crochet model, developed originally by Daina Taimina, is an approximation of hyperbolic geometry. The actual annulus model is the limit as $\delta \to 0$, where δ is the width of the annuli, and is not able to be C^2 embedded in any Euclidean $\mathbb{R}^n$. You can see $\delta \neq 0$ in the crocheted annuli. Consider how the Gaussian curvature K, the product of the principal normal curvatures, is intended to demonstrate negative curvature everywhere on this model.

2. Fold the model to create intrinsically straight paths—the folding gives symmetry of our feet walking along it on either side—and explore how the angle sum of a hyperbolic triangle is less than π, just as Gauss-Bonnet gave us in the interactive video. Sketch a picture in your notes.

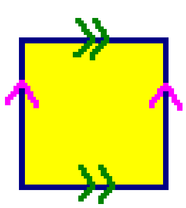
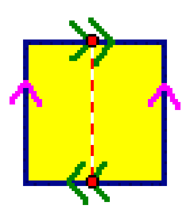
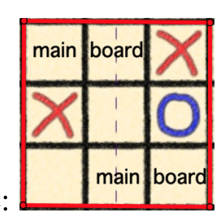
3. Given such a folded geodesic and a point off of this geodesic, how many parallel geodesics—-that don't intersect the first one—can you find? Discuss with your group and sketch a picture. Then compare with the usual parallel postulate as well as behavior on the $\frac{5\pi}{2}$ cone and respond on `pollev.com/drsarah314`

   a) There is exactly one parallel to a given geodesic through a point off of it, just as in the usual parallel postulate
   b) There is more than one parallel, just as in the $\frac{5\pi}{2}$ cone
   c) There are no parallels

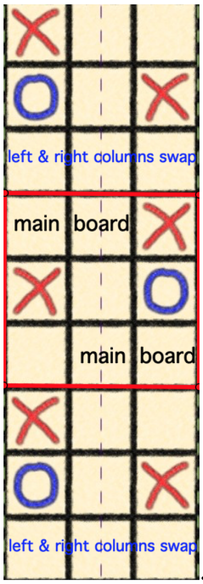
4. Review our prior work on the flat torus in $\mathbb{R}^4$ from the First Fundamental Form: Flat and Round Donuts worksheet. Recall that the flat torus can't be C^2 embedded in $\mathbb{R}^3$ but it can be in $\mathbb{R}^4$.

5. The flat Klein bottle can be embedded in $\mathbb{R}^5$ without intersections via $x(u,v) = (\cos u \cos v, \sin u \cos v, 2\cos\frac{u}{2}\sin v, 2\sin\frac{u}{2}\sin v, \cos v)$. The metric form can be calculated via $\mathbb{R}^5$ dot products of $E = \vec{x}_u \cdot \vec{x}_u$ and similar, giving $\begin{bmatrix} 1 & 0 \\ 0 & 2\cos^2 v + 2 \end{bmatrix}$ and we can calculate K intrinsically, like from Brioschi's intrinsic formula for K or an intrinsic formula for K when $F = 0$, like $K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right)$. Apply this last formula to the metric form. What is K here?

6. The flat torus and flat Klein bottle can be covered by identifying edges of a square, straight across for the torus, and with one twist for the Klein bottle:

   flat Torus:  flat Klein bottle:  Klein bottle Tic-Tac-Toe: 

   To the left and right in the covering, the flat Klein bottle is just like the flat torus, but the behavior is different above and below. Above on the right, in Klein bottle Tic-Tac-Toe from Jeff Weeks' Torus Games, we see a main board. Discuss with your group as you make sense of the following visualization of what happens above and below the main board: Then sketch the behavior of a vertical geodesic that is not the dotted line shown in the board but is shown inside these vertical sheets in the covering:



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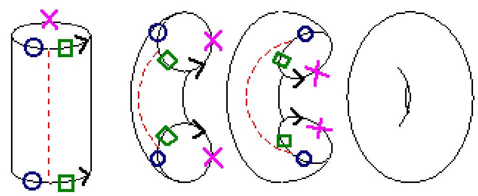
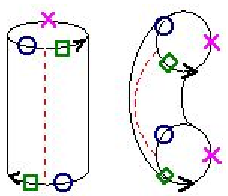
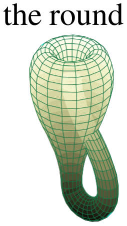
7. We can look at nonflat versions in $\mathbb{R}^3$, the round torus:  and the round Klein bottle:   CC-By-2.5 by Tttrung. The round Klein bottle has a nasty intersection when the slinky passes through itself. Our gluing instructions give no hint of this—the intersections only arose when we tried to glue the space in three dimensions, picking up distortions. I have placed a slinky and a Klein bottle model at your table so that you can better internalize the visualization. What kind of Gaussian curvatures are possible on this round Klein bottle?

8. In linear algebra section 6.1, which is a required section in that class, we explored inner products briefly. In general, as long as we have a matrix that is nonsingular, i.e. with the determinant not 0, and is symmetric, where $A = A^T$, i.e. if we reflect the matrix about the main diagonal of the matrix, swapping every row with every column to obtain the transpose, then we obtain the same matrix we started with, then we can use that for our metric form. The book reading for today explored examples of 2×2 matrices. For spacetime, we'll look at 4×4 matrices. Would this 4×4 matrix work $\begin{bmatrix} -1 + \frac{r_0}{r} - \frac{\epsilon}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_0}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$ where $r \neq 0$, i.e. is it symmetric with nonzero determinant? Respond on `pollev.com/drsarah314`

   a) yes
   b) no

9. For today, you began the in-class surfaces assessment guide. Discuss with your group and review.