Constant Curvature 1 Surfaces and Orbifolds

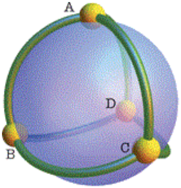
Gauss Curvature $K = \kappa_1 \kappa_2 = \frac{ln - m^2}{EG - F^2}$
http://homepage.math.uiowa.edu/~wseaman/DGImage53100.htm#Theorema%20Egregium1
Walter Seaman

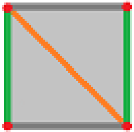
Gauss-Bonnet (compact smooth orientable, no boundary)

$$\int\int KdA = 2\pi\chi$$

$K = \kappa_1\kappa_2 = \frac{ln-m^2}{EG-F^2}$
dA=area
χ=Euler characteristic = Vertices - Edges + Faces

$\int\int KdA$	$2\pi\chi$
geom (K depends on shape)	topological (shape invariant)
curvature	combinatorics (counting)





2.bp.blogspot.com/-osMu4jpH7xo/VOHC9QgStWI/AAAAAAAAAkA/LhriHHymEA8/s1600/tetra.gif

- Triangulate with F geodesic triangular faces $\Delta_1, ...\Delta_F$
- Count V vertices and E edges.



http://cdn.sansimera.gr/media/photos/main/Pierre_Ossian_Bonnet.jpg, Impossiball,

Portrait by S. Bendixen 1828

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- Let $\alpha_i, \beta_i, \gamma_i$ be the angles of Δ_i
  total Gaussian K $= \int\int KdA$ = total angle defect of $\sum \Delta_i$
  $= \sum_i(\alpha_i + \beta_i + \gamma_i - \pi) =$

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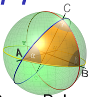
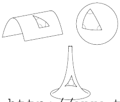
  $= \sum_i(\alpha_i + \beta_i + \gamma_i - \pi) = \sum_i(\alpha_i + \beta_i + \gamma_i) - F\pi$
- At each vertex V, the sum of the angles is 2π, so all the angles sum to $2\pi V$:
  $\int\int KdA = 2\pi V - F\pi = 2\pi(V - \frac{F}{2}) = 2\pi\chi$

Applications of Gauss-Bonnet





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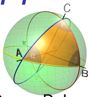

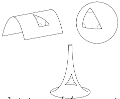

- $\int \int_{geod \Delta} KdA = (\alpha + \beta + \gamma - \pi)$
- $\int_{\partial S} \kappa_g ds + \int \int_S KdA = 2\pi\chi$

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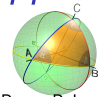

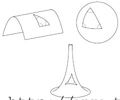

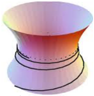

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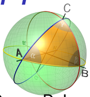
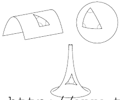
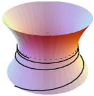
Applications of Gauss-Bonnet



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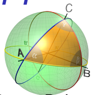
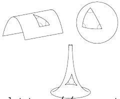
Applications of Gauss-Bonnet





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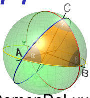
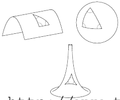
Applications of Gauss-Bonnet





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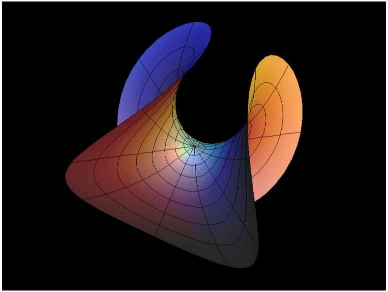
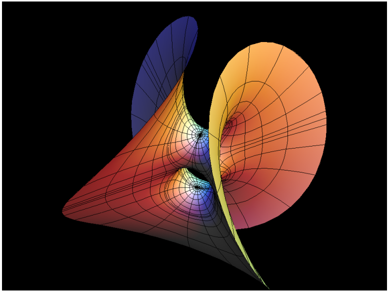
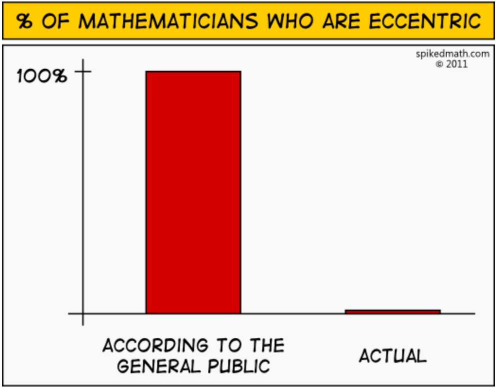
- $\int \int KdA = 2\pi\chi = 2\pi(2 - 2g) < 0$, where g genus, $\#$ holes

adding a hole changes the total curvature by -4π
Example: add a handle to Enneper's surface -4π to form Chen-Gackstatter surface -8π

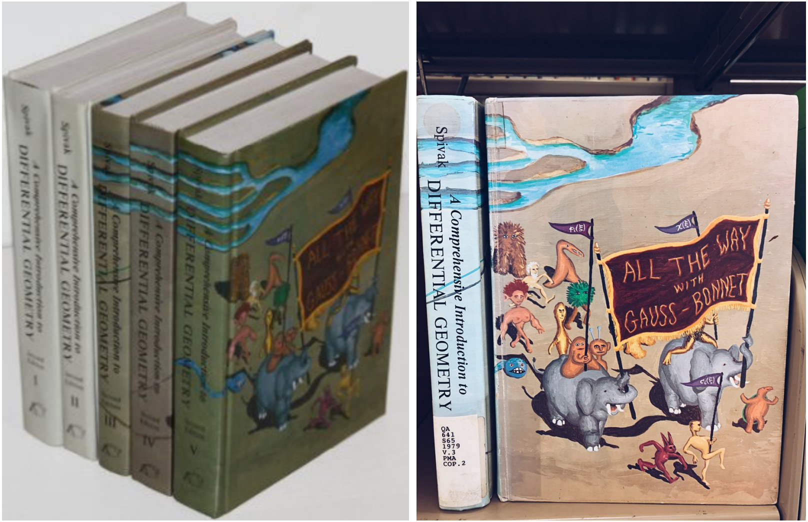




http://virtualmathmuseum.org/Surface/enneper/i/enneper2_polar.png, http://virtualmathmuseum.org/Surface/chen_gackstatter/i/chen_gackstatter_2-fold_88690.png

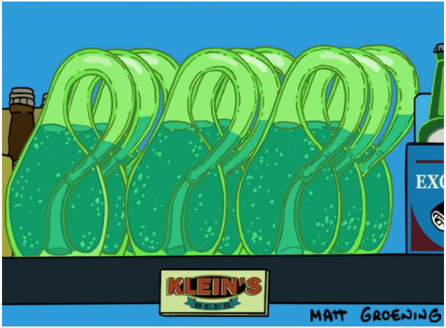


- Gauss published a special case of Gauss-Bonnet
$$\int\int_{geod\Delta} KdA = (\alpha + \beta + \gamma - \pi)$$
- Bonnet introduced the concepts of geodesic curvature and torsion, and published a more general version of Gauss-Bonnet



Michael Spivak *A Comprehensive Introduction to Differential Geometry*, Volume 5 cover

Surfaces Not in $\mathbb{R}^3$*: Klein's Beer*



Futurama: The Route of All Evil

Brioschi's K in higherdimcurvatures.mw and $F = 0$ formula

$$\frac{1}{(EG-F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right)$$