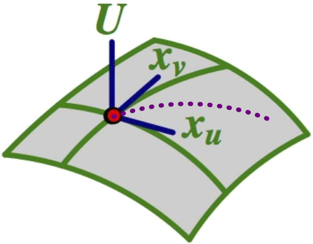
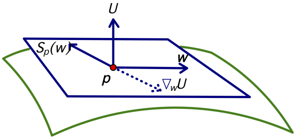
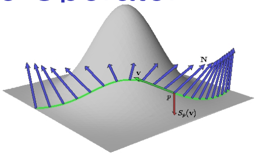
1st Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}, F = \vec{x_u} \cdot \vec{x_v}, G = \vec{x_v} \cdot \vec{x_v}$

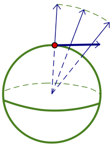


- Regular surface $M = \mathbf{x}(u, v)$, where $\vec{x}_u \times \vec{x}_v \neq 0$, and $u(t)$ & $v(t)$ give curve $\alpha(t)$. Then $\vec{x}_u, \vec{x}_v$ form basis for T_pM and
  $\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$
  $(\frac{ds}{dt})^2 = |\alpha'(t)|^2 = \alpha'(t)\cdot\alpha'(t) = (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})\cdot(\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})$
  $= \vec{x_u}\cdot\vec{x_u}(\frac{du}{dt})^2 + 2\vec{x_u}\cdot\vec{x_v}\frac{du}{dt}\frac{dv}{dt} + \vec{x_v}\cdot\vec{x_v}(\frac{dv}{dt})^2$
  $= E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$
  $ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^idu^j$

Shape Operator







2nd picture: The Center of Population of the United States
http://www.ams.org/publicoutreach/feature-column/fcarc-population-center

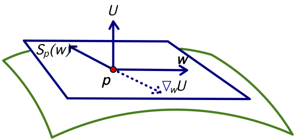
- curve: κ, τ rate of change of unit vector fields T & B ($\therefore$ N).
- surface: U unit vector field. Whole plane of directions—rates of change of U are measured, not numerically, but by a linear operator called the shape operator, which captures the bending of a surface.
- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$ compute for plane $\mathbf{x}(u, v) = (u, v, 0)$
  $\vec{x}_u, \vec{x}_v, U$
  $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
  $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
  any other $\vec{w} = a\vec{x}_u + b\vec{x}_v$ and $S_p(\vec{w}) = a\_\vec{x}_u + b\_\vec{x}_v$

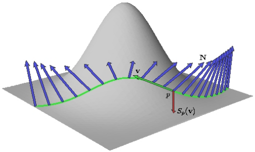
$S_p(\vec{w}) = -\nabla_{\vec{w}} U$ for geographical sphere

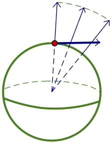
$\mathbf{x}(u, v) = (r \cos u \cos v, r \sin u \cos v, r \sin v)$

- $\vec{x}_u =$
- $\vec{x}_v =$
- $U =$
- $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$
  Then write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$
  Then write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
- any other $\vec{w} = a\vec{x}_u + b\vec{x}_v$ and $S_p(\vec{w}) = a\_\vec{x}_u + b\_\vec{x}_v$
  $\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

2nd Fundamental Form $l = \vec{x}_{uu} \cdot U, m = \vec{x}_{uv} \cdot U, n = \vec{x}_{vv} \cdot U$







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- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$ compute for plane & geographical sphere
- prove $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U$

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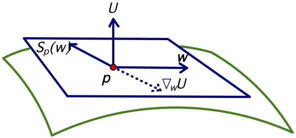

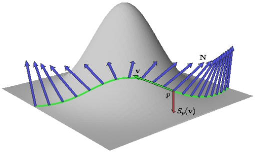

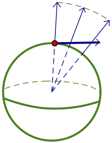

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- prove $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U$
  take derivative of $U \cdot \vec{x}_u = 0$ with respect to u

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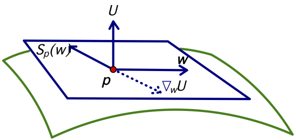

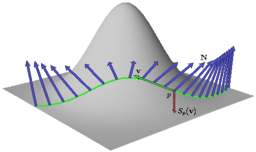

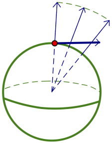

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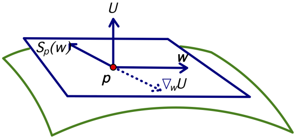
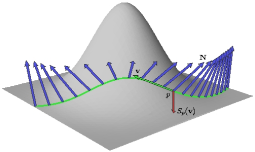
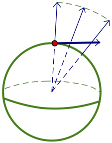
- prove $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U$
  take derivative of $U \cdot \vec{x}_u = 0$ with respect to u
  $0 = U_u \cdot \vec{x}_u + U \cdot \vec{x}_{uu}$

2nd Fundamental Form $l = \vec{x}_{uu} \cdot U, m = \vec{x}_{uv} \cdot U, n = \vec{x}_{vv} \cdot U$







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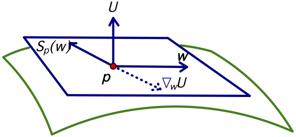
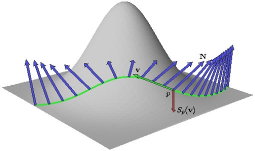
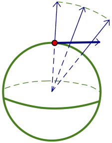
- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$ compute for plane & geographical sphere
- prove $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U$
  take derivative of $U \cdot \vec{x}_u = 0$ with respect to u
  $0 = U_u \cdot \vec{x}_u + U \cdot \vec{x}_{uu} = -S(\vec{x}_u) \cdot \vec{x}_u + U \cdot \vec{x}_{uu}$

2nd Fundamental Form $l = \vec{x}_{uu} \cdot U, m = \vec{x}_{uv} \cdot U, n = \vec{x}_{vv} \cdot U$







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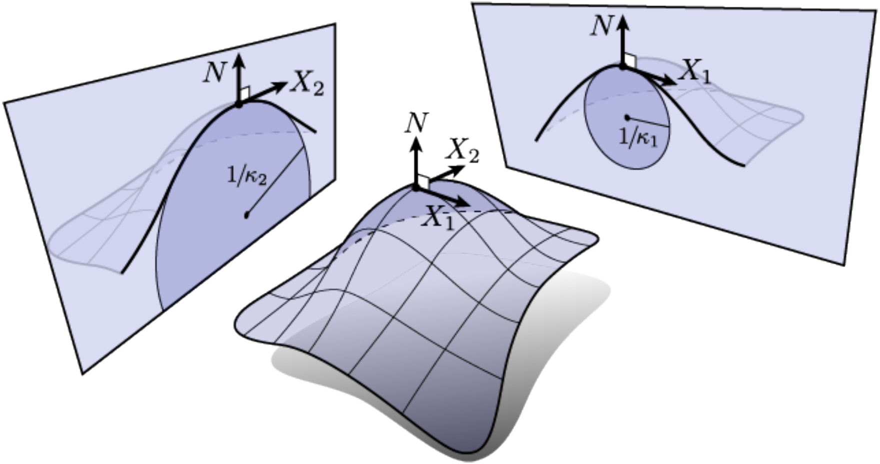
- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$ compute for plane & geographical sphere
- prove $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U = l$, $S(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_{uv} \cdot U = m$, $S(\vec{x}_v) \cdot \vec{x}_v = \vec{x}_{vv} \cdot U = n$ compute for geographical sphere
- eigenvalues of the shape operator: max and min normal curvature at p, called the principal curvatures κ_1 and κ_2

S Eigenvalues—Principal Curvatures κ_1 *and* κ_2



http://brickisland.net/cs177/?p=144

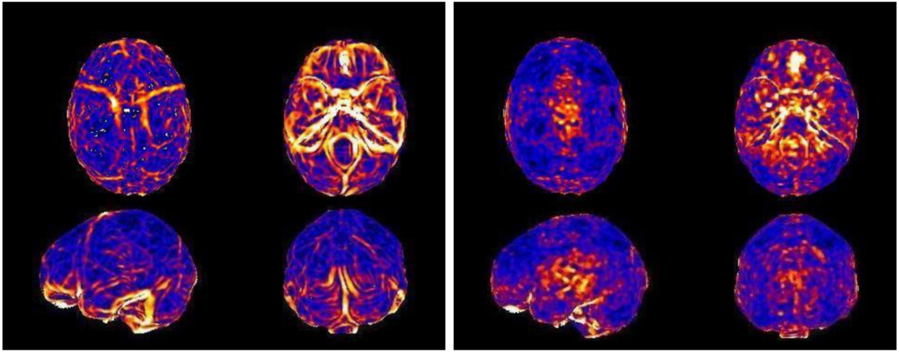
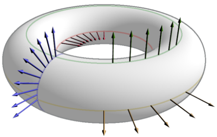
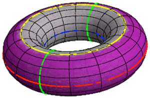


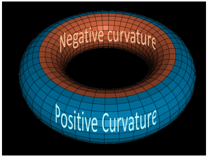
Fig. 3. Shape operator (left) and polynomial fit-derived (right) magnitude of curvature on the inner skull surface. Note that the shape operator is sensitive enough to assign high curvature to small structures, such as the vessel impressions on the inner skull surface. The polynomial-fit curvature image was processed with surface-constrained smoothing, to reduce noise, while the shape operator curvature did not require smoothing.

Avants, Brian and James Gee (2003) "The Shape Operator for Differential Analysis of Images," *Inf Process Med Imaging*. Jul 18:101–13.

Round Donut Shape Operator, K, H



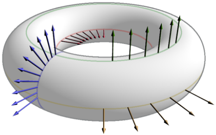


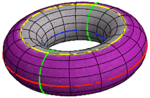


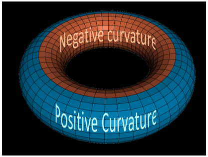
rdrop.com/~half/math/torus/shape.operator.xhtml i.stack.imgur.com/EbGem.png, Gregors CC-BY-SA-3.0

$$x(u,v) = ((R + r\cos u)\cos(v), (R + r\cos u)\cos(v)\sin(v), r\sin u)$$
$$\vec{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$
$$\vec{x}_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$$
$$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$

Round Donut Shape Operator, K, H







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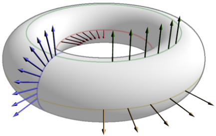
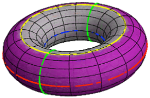
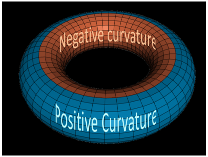
$$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$

$$-U_u = (\sin u\cos v, \sin u\sin v, -\cos u) = -(\_\_\vec{x}_u + \_\_\vec{x}_v) =$$

Round Donut Shape Operator, K, H







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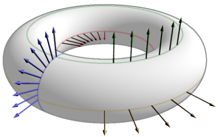
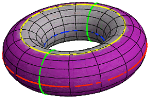
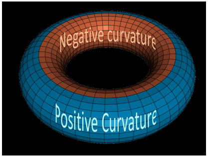
$$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$
$$-U_u = (\sin u\cos v, \sin u\sin v, -\cos u) = -(\_\_\vec{x}_u + \_\_\vec{x}_v) = \frac{1}{r}\vec{x}_u$$

Round Donut Shape Operator, K, H







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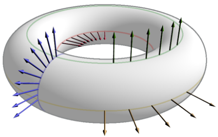
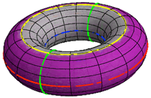
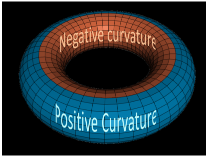
$$-U_v =$$

Round Donut Shape Operator, K, H







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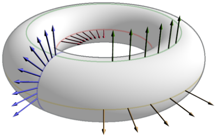
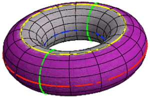
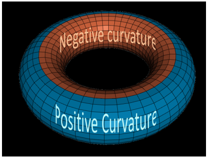
$$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$
$$-U_u = (\sin u\cos v, \sin u\sin v, -\cos u) = -(\underline{\quad}\vec{x}_u + \underline{\quad}\vec{x}_v) = \frac{1}{r}\vec{x}_u$$
$$-U_v = (\cos u\sin v, -\cos u\cos v, 0) = -(\underline{\quad}\vec{x}_u + \underline{\quad}\vec{x}_v) = \frac{\cos u}{R + r\cos u}\vec{x}_v$$
$$S(a\vec{x}_u + b\vec{x}_v) = \frac{a}{r}\vec{x}_u + \frac{b\cos u}{R + r\cos u}\vec{x}_v \qquad \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos u}{R + r\cos u} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

The eigenvalues are the principal (normal) curvatures κ_1, κ_2 and $K = \kappa_1\kappa_2 = \frac{\cos u}{r(R + r\cos u)}$, $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2}\left(\frac{1}{r} + \frac{\cos u}{R + r\cos u}\right)$

Gaussian curvature intuition for K

Round Donut 2nd Fundamental Form l, m, n

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$\vec{x}_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$

$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$

$U_u = (\sin u\cos v, \sin u\sin v, -\cos u) = \_\vec{x}_u + \_\vec{x}_v = -\frac{1}{r}\vec{x}_u$

$U_v = (\cos u\sin v, -\cos u\cos v, 0) = \_\vec{x}_u + \_\vec{x}_v = -\frac{\cos u}{R + r\cos u}x_v$

$l = S(\vec{x}_u)\cdot\vec{x}_u = -U_u\cdot\vec{x}_u =$

$r\sin^2 u\cos^2 v + r\sin^2 u\sin^2 v + r\cos^2 u = \vec{x}_{uu}\cdot U$

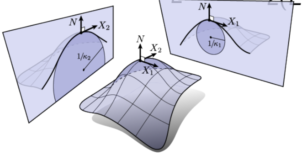
$m = S(\vec{x}_u)\cdot\vec{x}_v = \vec{x}_{uv}\cdot U$

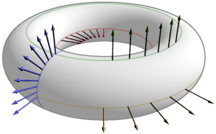
$n = S(\vec{x}_v)\cdot\vec{x}_v = \vec{x}_{vv}\cdot U$

Let's Get Bent: Gauss and Mean Curvature

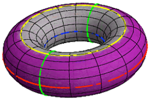
Gauss curvature: $K = \kappa_1\kappa_2 = \frac{ln-m^2}{EG-F^2} = \frac{|II|}{|I|}$ determinants

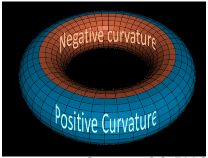
Mean curvature: $H = \frac{\kappa_1+\kappa_2}{2} = \frac{lG-2mF+nE}{2(EG-F^2)}$





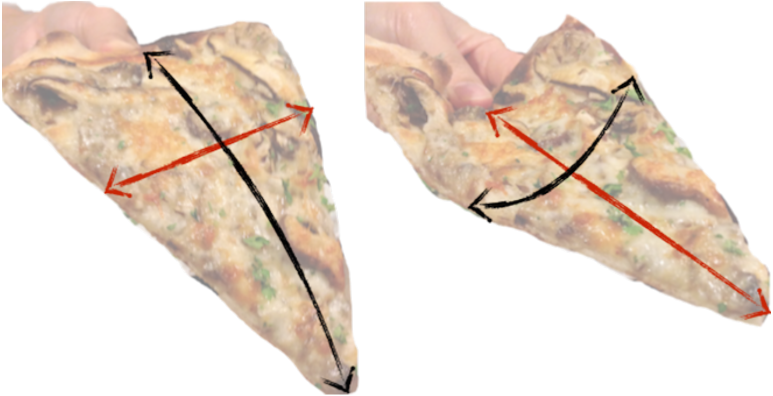
http://brickisland.net/cs177/?p=144, rdrop.com/~half/math/torus/shape.operator.xhtml





http://i.stack.imgur.com/EbGem.png,

- intuition with principal normal curvatures/tangent plane
- minimal surfaces, universe, medical imaging, population center
- plants produce curved or wrinkled leaves by altering the rate the edges of the leaf grow compared to the center.



https://www.wired.com/2014/09/curvature-and-strength-empzeal/

“How Carl Friedrich Gauss Taught Us the Best Way to Hold a Pizza Slice”

Gauss's Theorema Egregium

- The first fundamental form is intrinsic (E, F, and G) and can measure on the surface without knowledge of the embedding. The second fundamental form is extrinsic (l, m, and n) and helps describe how the surface is embedded in space as U changes.

Gauss's Theorema Egregium

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$$\frac{1}{(EG-F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix} \right)$$

1. If we have a surface $\mathbf{x}(u, v)$ and we hold u constant, then $\mathbf{x}(\text{constant}, v)$ will be a

a) surface and I have a good reason why

b) surface but I am unsure of why

c) curve but I am unsure of why

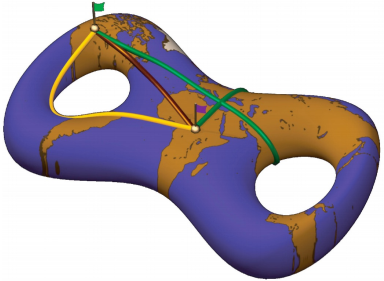
d) curve and I have a good reason why

e) other

2. On a round donut, sitting on a table, the following are geodesics:

a) all verticals on the donut
b) all horizonals on the donut
c) both of the above
d) none of the above

Next, think about a 2-holed donut:



http://page.mi.fu-berlin.de/polthier/video/Geodesics/Figures/ShrinkGeod_med.jpg

3. On a flat torus in $\mathbb{R}^4$, with the covering sitting in front of you on the table, the following are geodesics:

a) all verticals on the donut
b) all horizonals on the donut
c) both of the above
d) none of the above

Portrait of Gauss by S. Bendixen (1828)

Carl Friedrich Gauss Finally, I succeeded—not on account of my hard efforts, but by the grace of the Lord. Like a sudden flash of lightning... I am unable to say what was the conducting thread that connected what I previously knew with... success [Eves, 1972]

Sophie Germain This leads me to confess that I am not as completely unknown to you... but that fearing the ridicule attached to a female scientist, I have previously taken the name of M. LeBlanc in communicating to you... [Letter to Gauss, 1807]