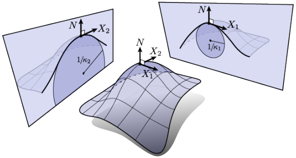
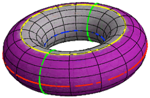
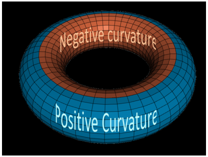
Let's Get Bent: Gauss and Mean Curvature

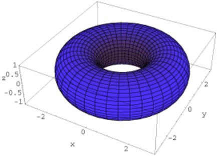
Gauss curvature: $K = \kappa_1 \kappa_2 = \frac{ln - m^2}{EG - F^2}$ Determinant

Mean curvature: $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{lG - 2mF + nE}{2(EG - F^2)}$ Trace









Applications:

- minimal surfaces, curvature of universe
- plants produce curved or wrinkled leaves by altering the rate the edges of the leaf grow compared to the center.

Gauss's Theorema Egregium

- The first fundamental form is intrinsic (E, F, and G) and can measure on the surface without knowledge of the embedding. The second fundamental form is extrinsic (l, m, and n) and helps describe how the surface is embedded in space as U changes.

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$$\frac{1}{EG-F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix} \right)$$

$$x(u,v) = ((R + r\cos u)\cos(v), (R + r\cos u)\cos(v)\sin(v), r\sin u)$$
$$x_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$
$$x_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$$
$$U = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$
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$$U_v = (\cos u\sin v, -\cos u\cos v, 0) = -\frac{\cos u}{R + r\cos u}x_v$$

$$S = \begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{\cos u}{R + r\cos u} \end{bmatrix}$$

The eigenvalues are the principal curvatures and we multiply them for

$K = \frac{\cos u}{r(R + r\cos u)}$.

Their average is the mean curvature H.