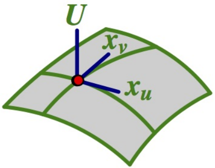
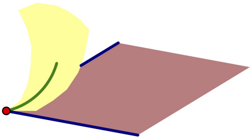
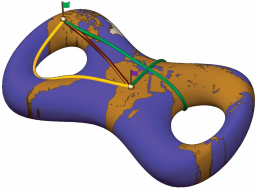
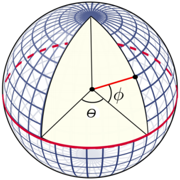
# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$








- Surface $\mathbf{x}(u, v)$ and curve $\alpha(t)$ on it given by $u(t)$ & $v(t)$.
  $\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$

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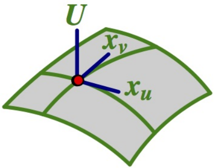


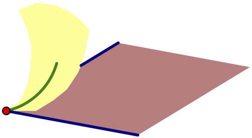

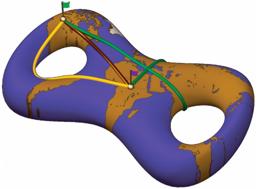

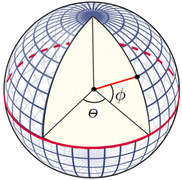


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  $\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$
  $(\frac{ds}{dt})^2 =$

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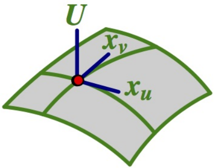
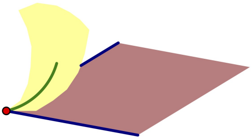
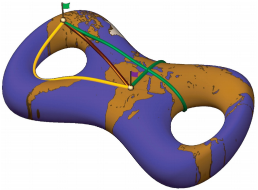
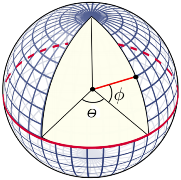


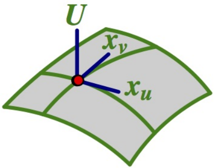



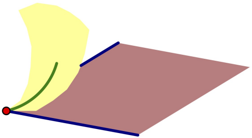


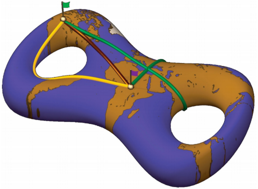


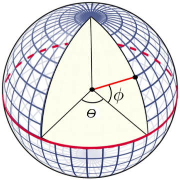

- Surface $\mathbf{x}(u, v)$ and curve $\alpha(t)$ on it given by $u(t)$ & $v(t)$.
  $\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$
  $(\frac{ds}{dt})^2 = |\alpha'(t)|^2 = \alpha'(t) \cdot \alpha'(t) = (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}) \cdot (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$







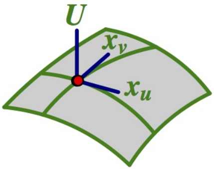

- Surface $\mathbf{x}(u, v)$ and curve $\alpha(t)$ on it given by $u(t)$ & $v(t)$.

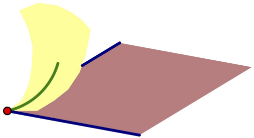
$\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$

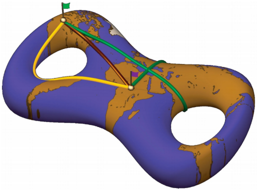
$(\frac{ds}{dt})^2 = |\alpha'(t)|^2 = \alpha'(t)\cdot\alpha'(t) = (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})\cdot(\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})$

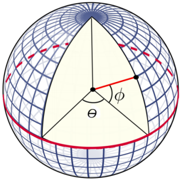
$= \vec{x_u} \cdot \vec{x_u}(\frac{du}{dt})^2 + 2\vec{x_u} \cdot \vec{x_v}\frac{du}{dt}\frac{dv}{dt} + \vec{x_v} \cdot \vec{x_v}(\frac{dv}{dt})^2$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$



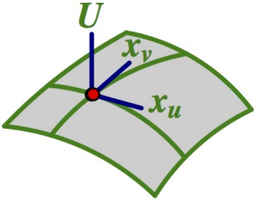



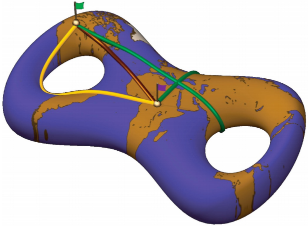


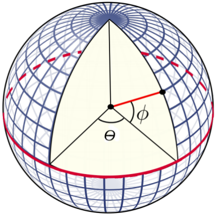



- Surface $\mathbf{x}(u, v)$ and curve $\alpha(t)$ on it given by $u(t)$ & $v(t)$.
  $\alpha'(t) = \vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}$
  $(\frac{ds}{dt})^2 = |\alpha'(t)|^2 = \alpha'(t) \cdot \alpha'(t) = (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt}) \cdot (\vec{x_u}\frac{du}{dt} + \vec{x_v}\frac{dv}{dt})$
  $= \vec{x_u} \cdot \vec{x_u}(\frac{du}{dt})^2 + 2\vec{x_u} \cdot \vec{x_v}\frac{du}{dt}\frac{dv}{dt} + \vec{x_v} \cdot \vec{x_v}(\frac{dv}{dt})^2$
  $= E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$
- E, F, G play important roles in many intrinsic properties of a surface like length, area and angles
- Example 1: $\mathbf{x}(u, v) = (u, v, 0)$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$

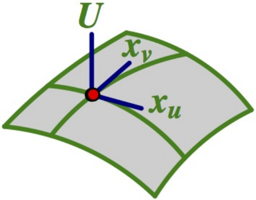


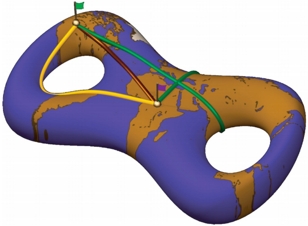



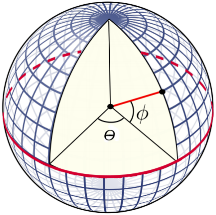



- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
- $g_{ij}$ determines dot products of tangent vectors

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$






- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
- $g_{ij}$ determines dot products of tangent vectors
- E, F, G play important roles in many intrinsic properties of a surface like length, area and angles

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

$$ds^2 = g_{11}du^1du^1 + g_{12}du^1du^2 + g_{21}du^2du^1 + g_{22}du^2du^2 = \sum_{i,j} g_{ij}du^idu^j$$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}, F = \vec{x_u} \cdot \vec{x_v}, G = \vec{x_v} \cdot \vec{x_v}$

- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
- $g_{ij}$ determines dot products of tangent vectors $\vec{w}_1, \vec{w}_2$ in $T_pM$

$\{\vec{x_u}, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$ , $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

$\vec{w}_1 \cdot \vec{w}_2 \overset{\text{foil}}{=}$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}, F = \vec{x_u} \cdot \vec{x_v}, G = \vec{x_v} \cdot \vec{x_v}$

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$\{\vec{x}_u, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$ , $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

$$\vec{w}_1 \cdot \vec{w}_2 \overset{\text{foil}}{=} ac\vec{x}_u \cdot \vec{x}_u + (ad + bc)\vec{x_u} \cdot \vec{x}_v + bd\vec{x}_v \cdot \vec{x}_v$$
$$= acE + (ad + bc)F + bdG$$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$

- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
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$\{\vec{x}_u, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$ , $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

$$\begin{aligned} \vec{w}_1 \cdot \vec{w}_2 &\overset{\text{foil}}{=} ac\vec{x}_u \cdot \vec{x}_u + (ad + bc)\vec{x_u} \cdot \vec{x}_v + bd\vec{x}_v \cdot \vec{x}_v \\ &= acE + (ad + bc)F + bdG = a(cE + dF) + b(cF + dG) \\ &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} cE + dF \\ cF + dG \end{bmatrix} \end{aligned}$$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}$, $F = \vec{x_u} \cdot \vec{x_v}$, $G = \vec{x_v} \cdot \vec{x_v}$

- Matrix representation: $g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
- $g_{ij}$ determines dot products of tangent vectors $\vec{w}_1$, $\vec{w}_2$ in $T_pM$

$\{\vec{x}_u, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$ , $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

$$\begin{aligned}
\vec{w}_1 \cdot \vec{w}_2 &\overset{\text{foil}}{=} ac\vec{x}_u \cdot \vec{x}_u + (ad + bc)\vec{x_u} \cdot \vec{x}_v + bd\vec{x}_v \cdot \vec{x}_v \\
&= acE + (ad + bc)F + bdG = a(cE + dF) + b(cF + dG) \\
&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} cE + dF \\ cF + dG \end{bmatrix} \\
&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}
\end{aligned}$$

# First Fundamental Form $E = \vec{x_u} \cdot \vec{x_u}, F = \vec{x_u} \cdot \vec{x_v}, G = \vec{x_v} \cdot \vec{x_v}$

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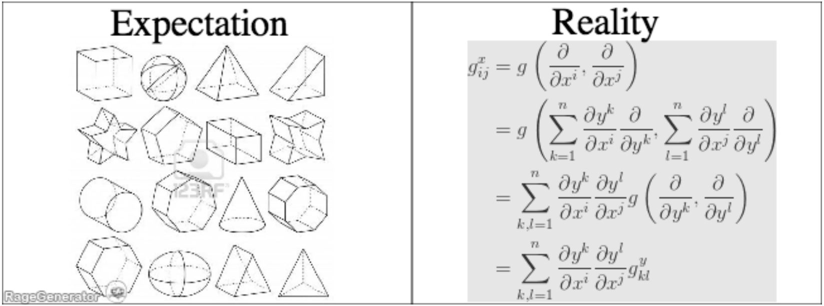
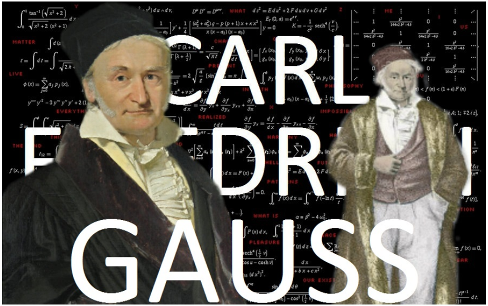
$\{\vec{x}_u, \vec{x_v}\}$ is a basis: $\vec{w}_1 = a\vec{x_u} + b\vec{x_v}$ , $\vec{w}_2 = c\vec{x_u} + d\vec{x_v}$

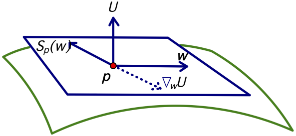
$$\vec{w}_1 \cdot \vec{w}_2 \overset{\text{foil}}{=} ac\vec{x}_u \cdot \vec{x}_u + (ad+bc)\vec{x_u} \cdot \vec{x}_v + bd\vec{x}_v \cdot \vec{x}_v$$
$$= acE + (ad+bc)F + bdG = a(cE+dF) + b(cF+dG)$$
$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} cE+dF \\ cF+dG \end{bmatrix}$$
$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$
$$= \vec{w}_1 \cdot \vec{w}_2 = |\vec{w}_1||\vec{w}_2| \cos\theta$$

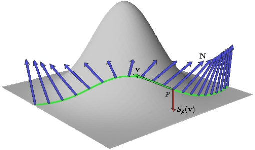
E, F, G play important roles in many intrinsic properties of a surface like length $(\frac{ds}{dt})^2$, area (det) and angles (above)


Expectation
Reality
$g^x_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$
$= g\left(\sum_{k=1}^{n} \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}, \sum_{l=1}^{n} \frac{\partial y^l}{\partial x^j}\frac{\partial}{\partial y^l}\right)$
$= \sum_{k,l=1}^{n} \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} g\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right)$
$= \sum_{k,l=1}^{n} \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} g^y_{kl}$
123RF
RageGenerator


CARL
GAUSS

# 2nd Fundamental Form $l = \vec{x}_{uu} \cdot U, m = \vec{x}_{uv} \cdot U, n = \vec{x}_{vv} \cdot U$

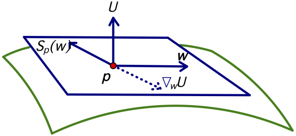


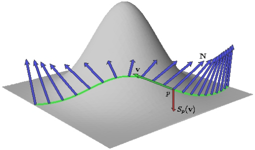




2nd picture: The Center of Population of the United States
http://www.ams.org/publicoutreach/feature-column/fcarc-population-center

- curve: $\kappa$, $\tau$ rate of change of unit vector fields T & B ($\therefore$ $N$).
- surface: $U$ unit vector field. Whole plane of directions—rates of change of $U$ are measured, not numerically, but by a linear operator called the shape operator, which captures the bending of a surface.

# 2nd Fundamental Form $l = \vec{x}_{uu} \cdot U, m = \vec{x}_{uv} \cdot U, n = \vec{x}_{vv} \cdot U$






2nd picture: The Center of Population of the United States
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- curve: $\kappa$, $\tau$ rate of change of unit vector fields T & B ($\therefore N$).
- surface: $U$ unit vector field. Whole plane of directions—rates of change of $U$ are measured, not numerically, but by a linear operator called the shape operator, which captures the bending of a surface.
- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$
- $S(\vec{x}_u) \cdot \vec{x}_u = \vec{x}_{uu} \cdot U = l$, $S(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_{uv} \cdot U = m$, $S(\vec{x}_v) \cdot \vec{x}_v = \vec{x}_{vv} \cdot U = n$
- eigenvalues of the shape operator: max and min normal curvature at $p$, called the principal curvatures $\kappa_1$ and $\kappa_2$