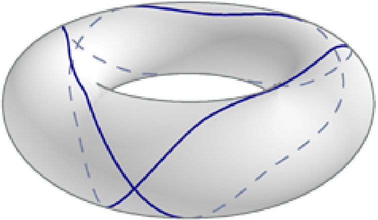
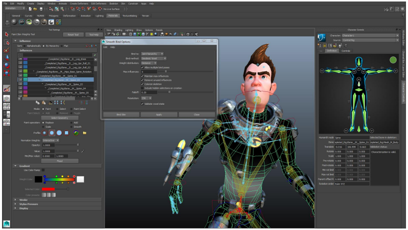
How do we find geodesics?





http://i.stack.imgur.com/La4Hj.png
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- covering
- symmetry
- proof on the sphere used the surface normal $U = \frac{\gamma}{R}$ in γ'' along with Frenet equations
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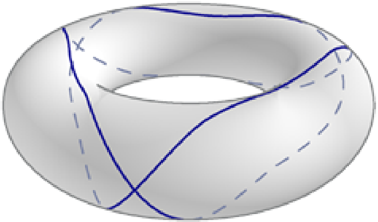
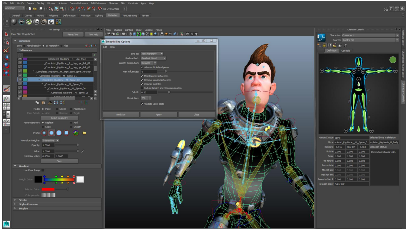

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- covering
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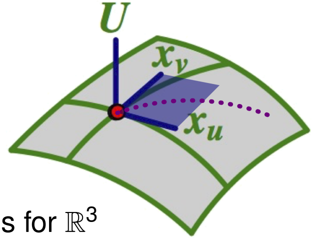
- how about more generally?
- guess and check in Maple or computer software
- parallel transport—that a tangent vector stays parallel
- geodesic equations with Christoffel symbols (also useful for Einstein's field equations)

Geodesic Equations on Surfaces in $\mathbb{R}^3$ with $F = 0$



$\{\vec{x}_u, \vec{x}_v, U\}$ is a basis for $\mathbb{R}^3$

α'' has to be only in the normal direction, so (eventually) write it in the basis and set $\vec{x}_u, \vec{x}_v$ components = 0.

$\alpha'(t) \overset{\text{chain rule}}{=} \vec{x}_u \frac{du}{dt} + \vec{x}_v \frac{dv}{dt}$

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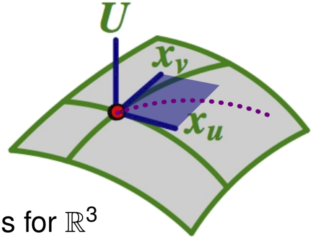

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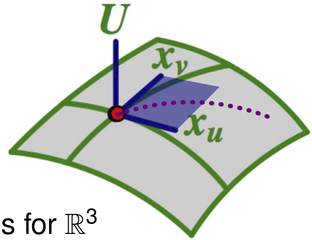

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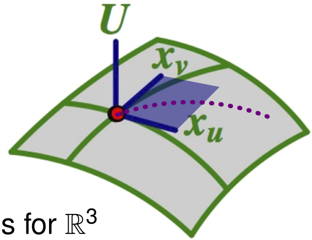

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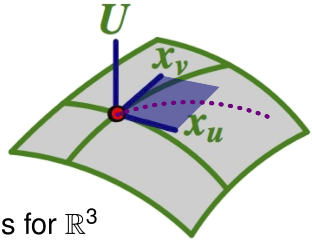

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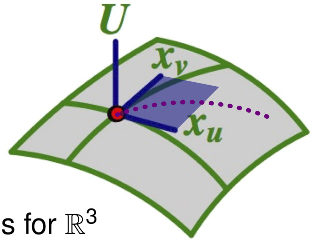

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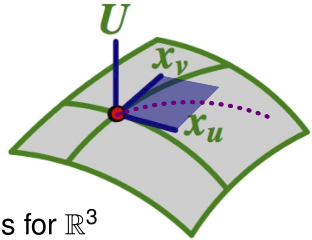

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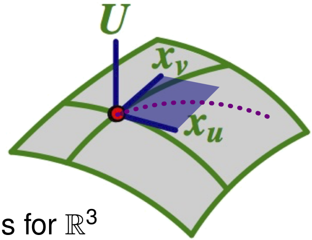

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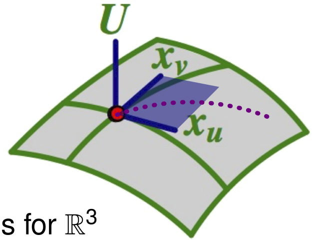

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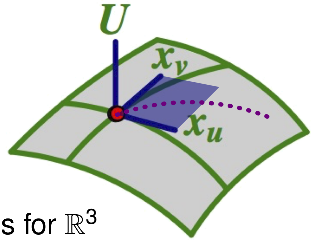

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Γ^c_{ab} called Christoffel symbols. $\vec{x}_{uv} = \Gamma^u_{uv}\vec{x}_u + \Gamma^v_{uv}\vec{x}_v + mU = \vec{x}_{vu}$.

$\vec{x}_{vv} = \Gamma^u_{vv}\vec{x}_u + \Gamma^v_{vv}\vec{x}_v + nU$

Geodesic Equations

set $\vec{x}_u, \vec{x}_v$ components = 0.

$\ddot{u} + \Gamma^u_{uu}\dot{u}^2 + 2\Gamma^u_{uv}\dot{v}\dot{u} + \Gamma^u_{vv}\dot{v}^2 = 0$ or $\ddot{x}^1 + \Gamma^1_{bc}\dot{x}^b\dot{x}^c = 0$

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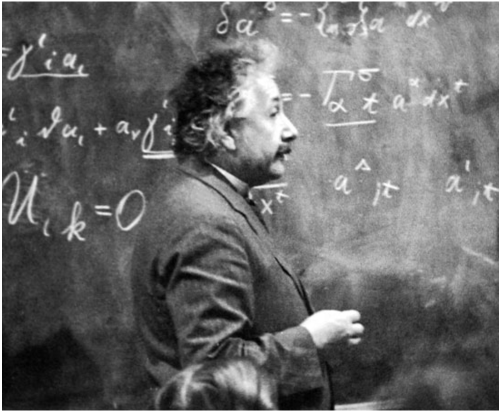
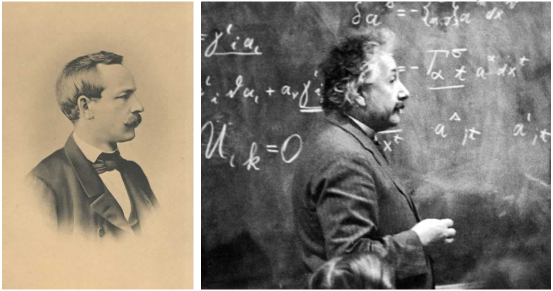
Or even more Einstein summation notation! $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$



http://scienceblogs.com/startswithabang/files/2013/07/einstein.jpg

How do we find the Christoffel symbols?



Elwin Bruno Christoffel and Albert Einstein

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$E_u =$

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 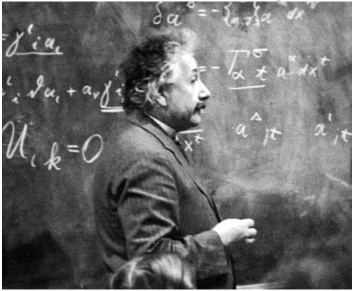

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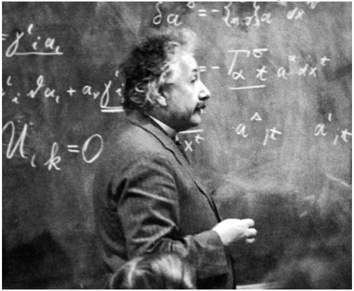

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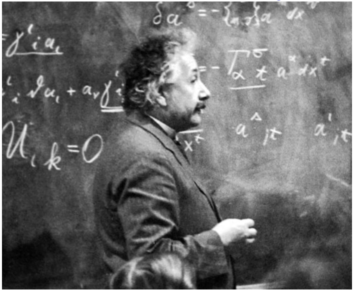

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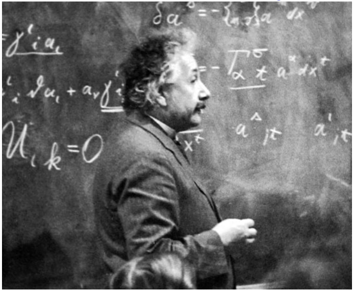

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Thus $\frac{E_u}{2} = \vec{x}_{uu} \cdot \vec{x}_u = \Gamma^u_{uu}E$ so $\Gamma^u_{uu} = \frac{E_u}{2E}$

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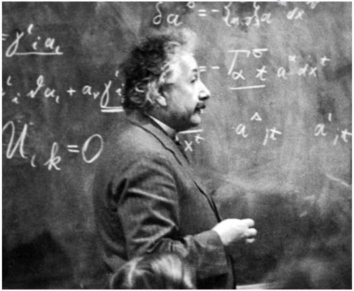

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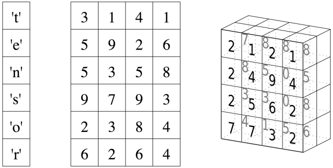
Similarly $\Gamma^v_{uu} = -\frac{E_v}{2G}$, $\Gamma^u_{uv} = \frac{E_v}{2E}$, $\Gamma^v_{uv} = \frac{G_u}{2G}$, $\Gamma^u_{vv} = -\frac{G_u}{2E}$, $\Gamma^v_{vv} = \frac{G_v}{2G}$

Solving the Geodesic Equations?

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

- differential equations expressed in intrinsic coordinates
- theoretical importance in mathematics and physics in analytic treatments of geodesics
- in practice, these equations can rarely be solved, except approximately (numerically)
- conetorusgeos.mw adapted from demo by John Oprea

In Higher Dimensions? Tensors and the Metric Tensor g_{ij}



towardsdatascience.com/a-beginner-introduction-to-tensorflow-part-1-6d139e038278

lists #s vectors stack of matrices

- algebraic combinations of vectors, matrices, vector spaces, algebras, modules or other structures
- often geometrically meaningful
- not all tensors are inherently linear maps

g_{ij} inner products of tangent vectors $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

$$g^{ij} = g_{ij}^{-1} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

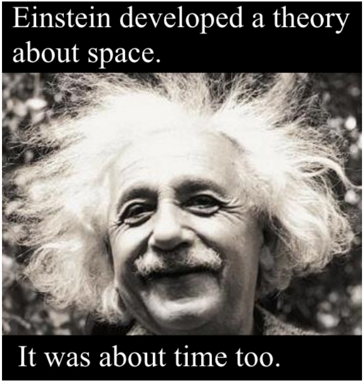
SpaceTime-Time: Special Relativity

Albert Einstein special relativity (1905)
Hermann Minkowski 4D spacetime model (1908)
surfaces: g_{ij} inner products of tangent vectors $w^T g_{ij} v$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

spacetime: metric tensor is now 4x4 symmetric matrix acting as $w^T g_{ij} v$ on (t, x, y, z) vectors. Yardstick plus clock!

Minkowski SpaceTime Model



http://jokerific.com/wp-content/uploads/2014/08/einstein-space-time-theory-joke.jpg

$$\begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$c, c^2, -$

In Higher Dimensions? Keep on Summing!

metric form

$$ds^2 = g_{ab}dx^a dx^b$$

Christoffel symbols

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}).$$

Riemann curvature tensor or Riemann-Christoffel tensor
$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$
Ricci tensor $R_{ab} = R^c_{acb} = g^{cd}R_{dacb}$
Scalar curvature $R = g^{ab}R_{ab}$
Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$

Christoffel Symbols and Curvatures

The Christoffel symbols

- intrinsic quantities, how to take covariant derivatives
- coefficients of tangent vectors (connection coefficients)
- measure whether or not vectors are parallel transports
- in relativity, gravitational forces. geodesics and curvatures.

Riemann curvature tensor: measures how much a manifold is not flat via $4^4 = 256$ entries for spacetime.

Ricci tensor: trace (sum of diagonal elements) relates to the metric volume $\sqrt{det g_{ij}}$.

Scalar curvature: number. For surfaces—twice Gaussian curvature. For relativity—Lagrangian density.

Einstein tensor describes curvature of spacetime due to the presence of energy or mass, has zero divergence

SpaceTime-Time: Other SpaceTimes

http://www.spacetime-model.com/img/mass/einstein.jpg

4D Manifold, g_{ij}, curvature satisfy Einstein field equation
g_{ij} can be other 4x4 symmetric matrices acting as $w^T g_{ij} v$.
g_{ij} 1 eigenvalue > 0 and three eigenvalues < 0 at each point.
angle : $\cos\theta = \frac{w^T g_{ij} v}{|v||w|}$, spacetime interval: $|v| = \sqrt{v^T g_{ij} v}$

- change g_{ij} and change physical properties of spacetime
- astrophysicist begins with what we observe and tries to construct a metric that models it
- enter any metric, then use Einstein's field equation to read off the physical properties of the resulting universe