Homework 3: Curves Continued

You may work alone or in a group of up to 2 people and turn in one per group. Each group must write up their work in their own words and give any credit to others where it is due. You do not need to simplify your answers, but you do need to show by-hand computations. Try all the problems, but carefully write up to turn in this number:
4140 students complete 4 of the first 5 problems. Graduate students complete all 6. The Maple file listed below homework3maple.mw is on the class webpage.

The purpose of homework is to learn and practice computational strategies, concepts, and develop critical thinking and problem-solving skills, so you should try problems on your own. Feel free to talk to me or each other if you are stuck on this assignment, but be sure to acknowledge any sources including each other, except your group members or me. If you know how to do a problem and are asked for help, try to give hints rather than the solution.

1. **Exercise 1.4.7 by-hand and in Maple**

   a) Do exercise 1.4.7 on page 31, **by-hand and show work for your computations**. Keep everything with respect to t. Use these formulas:
   $\frac{ds}{dt} = |\alpha'(t)|$ and $T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{\frac{ds}{dt}}$
   $\vec{\kappa} = \frac{T'(t)}{\frac{ds}{dt}}$ because of the chain rule $\vec{\kappa} = \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{T'(t)}{|\alpha'(t)|}$
   The curvature κ is the length of $\vec{\kappa} = |\vec{\kappa}|$ and $N(t) = \frac{\vec{\kappa}}{|\vec{\kappa}|}$
   $B(t) = T \times N$
   τ: To calculate the torsion by-hand in this context, where we are not parametrized by arc length, note that $\frac{B'(t)}{|\alpha'(t)|} = \frac{B'(t)}{\frac{ds}{dt}} = \frac{dB}{dt}\frac{dt}{ds} = B'(s)$ (by chain rule), and we defined this as $-\tau N$.
   So compute $\frac{B'(t)}{|\alpha'(t)|}$ & compare it to N (they are multiples of each other) to find $-\tau$ and then τ.

   b) Do exercise 1.4.7 on page 31 **in Maple** by adapting the file homework3maple.mw and print out your commands and the Maple output.

   c) Some people define τ by $B'(s) = -\tau N$ like we did and others by $B'(s) = +\tau N$. Compare Maple's torsion to the by-hand work—does Maple match our torsion and thus our definition of it? Does Maple use plus or minus to define the torsion?

   d) Compare your Maple and by-hand responses and if there are any, resolve any differences in the computations. If they are all the same, say so.

2. **Mystery Curve**

   a) First use Maple to help you do (only) part 4 of exercise 1.3.11 on page 21. You can use similar commands as in Question 1, except you are better off using simplify(Torsion(curve2,s)); rather than simplify(Torsion(curve2,s),trig);. You will print your commands and the Maple output.

   b) Compare the Maple outputs and use them to show why the curvature and torsion are the same.

   c) For what s values is the curvature defined? For these values, is the curvature positive, negative, zero, or a mixture? Explain.

   d) Then continue with exercise 1.5.4 on page 36. See the hint in the back of the book on p. 443 and use this to look for meaning as to what kind of curve this is. Explain.

3. **Osculating Circle by-hand and in Maple**

   a) **By hand:** Find the radius and equation of the circle that best fits the curve $y = x^2$ at $x_0 = 0$. You can use the formula from Calc III and Analytic Geometry for the (scalar) curvature of a function y=f(x), i.e. $\kappa = \frac{f''(x_0)}{(1+f'(x_0)^2)^{\frac{3}{2}}}$ or you can take the longer route and parametrize by t and then calculate κ by computing $|\frac{dT}{ds}|$ evaluated at $t = 0$. After calculating κ, find the radius and center of the circle.

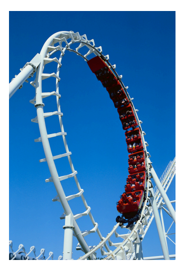
   b) Plot the curve and the circle in **Maple** on the same graph by adapting the file homework3maple.mw and print out your graph and commands.

4. **Frenet-Equations and the Darboux Vector**
   If a rigid body moves along the curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector $\omega(s)$, called the Darboux vector, named for Jean-Gaston Darboux. It satisfies the following:

$$T'(s) = \omega(s) \times T(s), \quad N'(s) = \omega(s) \times N(s), \quad B'(s) = \omega(s) \times B(s)$$

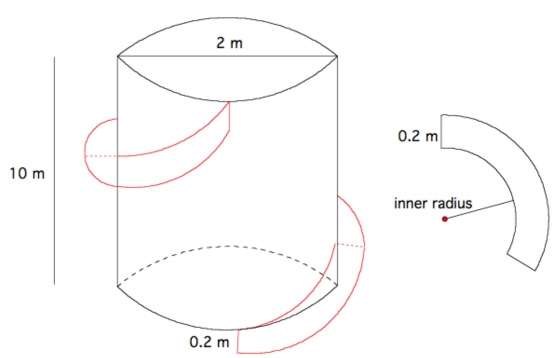
   a) Do exercise 1.3.12 on page 21.

   b) Write a short paragraph which interprets each term of $\omega = \tau T + \kappa B$ in the example of a roller coaster moving along α. (Think of the coaster car and the people inside as the rigid body. A good first example to consider is along a theoretical ferris wheel (circular) hump that stays in that ferris wheel plane (for a bit), where the car is on that part of the track and the front of the car is pointed in the direction of the tangent vector, like approximately in the picture below. Then consider the general case where torsion also comes in.)



   Include at least one rough sketch or a picture related to your explanations.

5. **Strake**



   a) Given the setup in the picture above, compute the ideal value for the inner radius of the annulus (your choice of method and your choice of by-hand or Maple to print).

b) Create a model of the annulus and the cylinder, approximately to scale. The model might be a real-life model or a printed Maple plot, for example.

c) A strake is not planar, but annular pieces of flat steel can sometimes be bent without stretching/stressing it too much to produce the strake. Other times a flat annulus could not be used. One can compare the local intrinsic geometry of the strake to the local geometry of the planar annulus as follows:

c1) Compute and show work (your choice of Maple commands to print and/or by-hand work, whatever you prefer here) for the curvatures and arc lengths of the inner and outer edges of the annulus and the corresponding inner and outer edges of the helical strake. So you will have 6 Maple or by-hand computations:

- inner annulus curvature = inner helix curvature
- inner annulus arc length = inner helix arc length
- outer annulus curvature
- outer annulus arc length
- outer helix curvature
- outer helix arc length

Hint: To calculate the length of the outer annulus, we can set up a proportion: the length of the inner annulus/its radius is the length of the outer annulus/its radius. The curvature of the outer annulus is 1 over its radius.

c2) Do the outer annulus and outer helix computations agree?

c3) What happens if we make the strake very wide compared to the diameter of the cylinder, such as in an auger below. Do you think this be made physically from a flat annulus? Explain why or why not.

6. **Graduate Problem**: Curve Proof
Let α be a curve with $\kappa > 0$. Show that if α's osculating planes have a point in common, call it p (i.e. each plane passes through p), then α is planar.