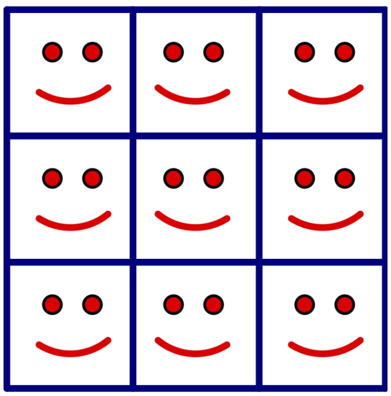
Homework 5: Flat Donuts and Round Donuts

You may work alone or in a group of up to 2 people and turn in one per group. Each group must write up their work in their own words. Feel free to talk to me or each other if you are stuck on this assignment, but be sure to acknowledge any sources including each other, except your group members or me. If you know how to do a problem and are asked for help, try to give hints rather than the solution. **Be sure to define all terms you use, draw or create your own pictures, explain in your own words, and give any references you used, if any, including each other.**

1. A Flat Donut

   (a) A flat torus can be theoretically obtained by taking a square and identifying the edges straight across (top to bottom and separately left to right) without distorting the geometry of the interior. So a covering space would be infinitely squares next to each other which are exact copies of each other. How many geodesics join two points in a flat torus? Explain in your own words and through pictures you draw or create yourself in the covering space.

   (b) Can a geodesic on a flat torus ever intersect itself by closing back up at a 0° or 180° angle? If so give an example and if not, explain. Explain in your own words and through pictures you draw or create yourself in the covering space.

   (c) Can a geodesic on a flat torus ever intersect itself in angles that differ from integer multiples of π? If so give an example and if not, explain. Explain in your own words and through pictures you draw or create yourself in the covering space.

   (d) A flat torus cannot be isometrically embedded in $\mathbb{R}^3$ as a smooth C^2 surface without distorting the curvature and geodesics (see problem 2). However, it can be embedded in $\mathbb{R}^4$ by an isometry so that the local intrinsic experience and geometric properties are the same. Read Example 5.4.9 beginning on p. 229, which discusses a parametrization of a flat torus in $\mathbb{R}^4$. In this parametrization, calculate $X_u, X_v, E = X_u \cdot X_u, F = X_u \cdot X_v$ and $G = X_v \cdot X_v$ by-hand and show work.

   (e) Next write down the metric form $(\frac{ds}{dt})^2$ and compare it with the Pythagorean theorem to explain why this is called "flat."

   (f) **Grad Problem** There is a C^1 isometric embedding of a flat torus in $\mathbb{R}^3$. Read pages 7–9 (Isometric embeddings: from Schlaefli to Nash, but stop at The Gromov Convex Integration Theory) of Borrelli, Vincint, Säıd Jabrane, Francis Lazarus, and Boris Thibert (2013). "Isometric embeddings of the square flat torus in ambient space." *Ensaios Matemáticos* 24 pp. 1–91, which is available at `https://www.emis.de/journals/em/images/pdf/em_24.pdf`. Summarize what Borelli et al. say about why C^1 is possible while C^2 is impossible, and what they say about curvatures and the Gauss map.

2. A Round Donut

(a) First read p. 73 Exercise 2.1.13, which gives a parametrization of a round torus. Open up the Maple file **curvature2.mw** on geodesic and normal curvatures that we have been using in class and input a parametrization for a round donut, choosing specific (nonzero) values for r and R. Next find a curve that is NOT a geodesic. Provide your new values for each of the following (these are the commands I used for the sphere–modify them for the torus) for a curve on a round donut that is NOT a geodesic on the round torus:

```
g := (x,y) -> [cos(x)*sin(y), sin(x)*sin(y), cos(y)]:
a1:=0: a2:=Pi: b1:=0: b2:=Pi:
c1 := 1: c2 := 3:
Point := 2:
f1:= (t) -> t:
f2:= (t) -> 1:
```

(b) Either print out or sketch by-hand a picture of the curve that is NOT a geodesic, the curvature vectors (label them to clarify which is which), and the torus.

(c) Now repeat the same process, but find a curve on a round donut that is a geodesic. Provide your new values for the Maple file, like you did in Part (a).

(d) Either print out or sketch by-hand a picture of the curve that is a geodesic, the curvature vectors (label them to clarify which is which), and the torus.

(e) Discuss your curves from an intrinsic point of view—i.e. why is the geodesic straight from the point of view of a bug, and why is the other curve not. You should refer to an intrinsic argument like the symmetries of your feet, a car that does or does not need to turn the steering wheel and/or whether narrow tape or a ribbon can lie on the curve without buckling.

(f) Examine the the roles of u and v in the book's parametrization of the donut: hold one constant and explain what kind of curve the other gives, and then the reverse.

(g) Use the Maple document on the first fundamental form **firstfundformhw.mw** to input the general round torus (leave r and R as variables), and then to calculate X_u, X_v and $X_u \times X_v$. Write down the results from Maple.

(h) Using Part (g), first explain why $X_u \times X_v$ is never the 0 vector.

(i) Next, what does Part (h) tell you about the existence of the tangent plane? Does it always, never or only sometimes exist for this parametrization. Explain.

(j) Use X_u and X_v from your Maple work in Part (g) to compute <u>by-hand</u> $F = X_u \cdot X_v$ and show work.

(k) Then interpret F. What does it tell you about the relationship between X_u and X_v?

(l) Use the same Maple document on the first fundamental form **firstfundformhw.mw** to calculate the matrix $EFG(X)$ and the determinant and write down what Maple provides.

(m) Next write down the metric form $(\frac{ds}{dt})^2$ and compare it with the Pythagorean theorem to show how it differs from a "flat" metric.

(n) **<u>Grad Problem</u>** Set up a double integral of the the square root of the determinant of the first fundamental form to find the surface area of the torus. Include the limits of integration and then integrate to solve. (you can compare with 2.29 on `https://cs.nyu.edu/~ajsecord/shells_course/html/shells_course-node10.html` about the First Fundamental Form)