Shape Operator

2nd picture: The Center of Population of the United States
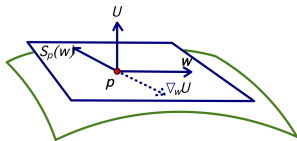http://www.ams.org/publicoutreach/feature-column/fcarc-population-center

- curve: κ, τ rate of change of unit vector fields T & B ($\therefore$ *N*).
- surface:
    - metric form g_{ij} tells us about behavior in the tangent plane
    - *U* unit vector field. Whole plane of directions—rates of change of *U* are measured, not numerically, but by a linear operator called the shape operator

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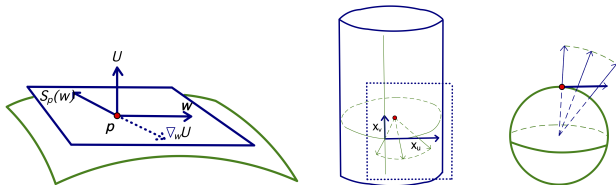
- $S_p(\vec{w}) = -\nabla_{\vec{w}} U$
- $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u = \_\vec{x}_u + \_\vec{x}_v$
- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v = \_\vec{x}_u + \_\vec{x}_v$
  Example: $x(u, v) = (u, v, 0)$

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Shape Operator of Cylinder

- $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$

Example: $x(u, v) = (r \cos u, r \sin u, v)$, with r constant

$x_u = (-r \sin u, r \cos u, 0)$
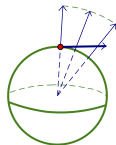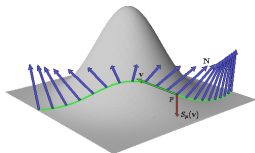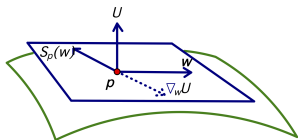
$x_v = (0, 0, 1)$

$U = (\cos u, \sin u, 0)$

$S(\vec{x}_u) = -U_u = -(-\sin u, \cos u, 0) = -\frac{1}{r} x_u + 0 x_v$

$S(\vec{x}_v) = -U_v = (0, 0, 0) = 0 x_u + 0 x_v$

$\begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix}$
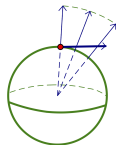
Shape Operator of Sphere

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- $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
  Example: $x(u, v) = (r \cos u \cos v, r \sin u \cos v, r \sin v)$

Shape Operator of Sphere
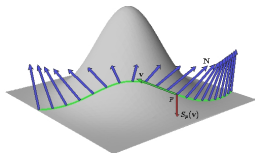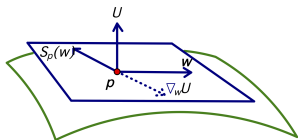
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  Example: $x(u, v) = (r \cos u \cos v, r \sin u \cos v, r \sin v)$
  $S(\vec{x}_u) = -U_u = -\frac{1}{r} x_u + 0 x_v$
  $S(\vec{x}_v) = -U_v = (0, 0, 0) = 0 x_u - \frac{1}{r} x_v$
  $\begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{bmatrix}$
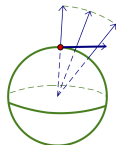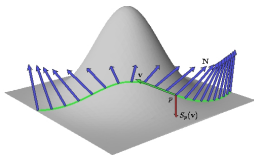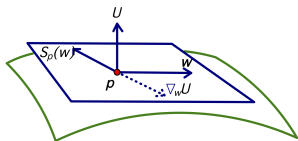
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Shape Operator of Psuedosphere

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- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\vec{x}_u + \_\vec{x}_v$
  Example: $x(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, u - \tanh u)$

Shape Operator of Psuedosphere
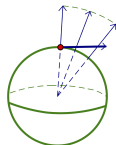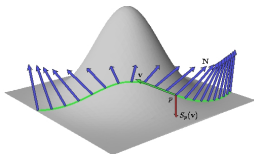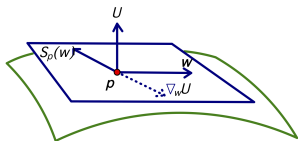
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  Example: $x(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, u - \tanh u)$
  $S(\vec{x}_u) = -\frac{1}{\sinh u}\vec{x}_u + 0\vec{x}_v$
  $S(\vec{x}_v) = 0\vec{x}_u + \sinh u\,\vec{x}_v$
  $\begin{bmatrix} -\frac{1}{\sinh u} & 0 \\ 0 & \sinh u \end{bmatrix}$

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Shape Operator of $u^2 - v^2$ Saddle

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- $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -U_u$ write it in the basis of $\_\_\vec{x}_u + \_\_\vec{x}_v$
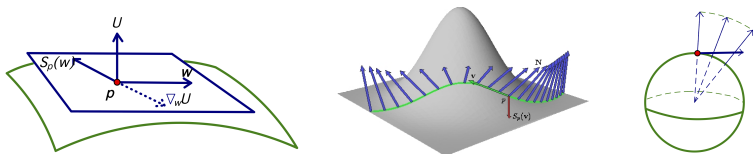- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\_\vec{x}_u + \_\_\vec{x}_v$
  Example: $x(u, v) = (u, v, u^2 - v^2)$

Shape Operator of $u^2 - v^2$ Saddle
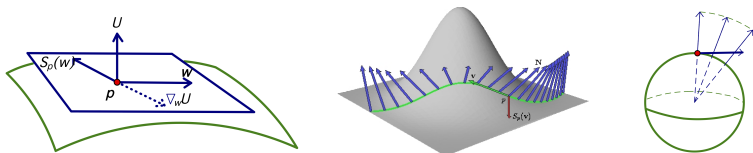
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- $S(\vec{x}_v) = -\nabla_{\vec{x}_v} U = -U_v$ write it in the basis of $\_\_\vec{x}_u + \_\_\vec{x}_v$

  Example: $x(u, v) = (u, v, u^2 - v^2)$

  $S(\vec{x}_u) = \frac{8v^2+2}{(1+4u^2+4v^2)^{\frac{3}{2}}} \vec{x}_u + \frac{8vu}{(1+4u^2+4v^2)^{\frac{3}{2}}} \vec{x}_v$

  $S(\vec{x}_v) = \frac{-8vu}{(1+4u^2+4v^2)^{\frac{3}{2}}} \vec{x}_u + -\frac{8u^2+2}{(1+4u^2+4v^2)^{\frac{3}{2}}} \vec{x}_v$

  $$\begin{bmatrix} \frac{8v^2+2}{(1+4u^2+4v^2)^{\frac{3}{2}}} & \frac{8vu}{(1+4u^2+4v^2)^{\frac{3}{2}}} \\ \frac{-8vu}{(1+4u^2+4v^2)^{\frac{3}{2}}} & -\frac{8u^2+2}{(1+4u^2+4v^2)^{\frac{3}{2}}} \end{bmatrix}$$
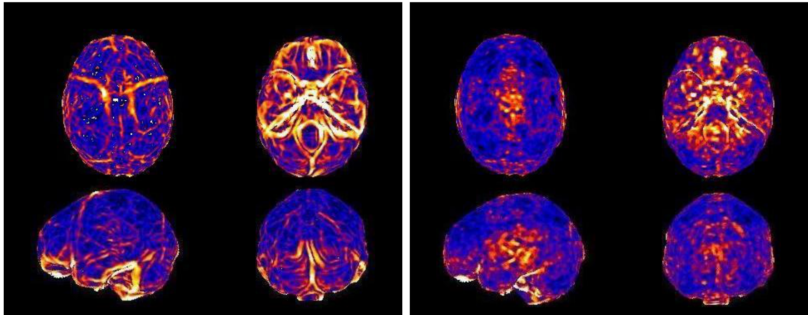
Fig. 3. Shape operator (left) and polynomial fit-derived (right) magnitude of curvature on the inner skull surface. Note that the shape operator is sensitive enough to assign high curvature to small structures, such as the vessel impressions on the inner skull surface. The polynomial-fit curvature image was processed with surface-constrained smoothing, to reduce noise, while the shape operator curvature did not require smoothing.

Avants, Brian and James Gee (2003) "The Shape Operator for Differential Analysis of Images," *Inf Process Med Imaging*. Jul 18:101–13.