Use of Euler Characteristics in the Classification of Surfaces

Matthew J. Mellon

December 8, 2001

Homeomorphism

Definition. [Homeomorphism] *Let* (X, T) *and* (Y, Υ) *be topological spaces. A bijection* $f : X \to Y$ *is a* **homeomorphism** *provided that* f *and* f^{-1} *are continuous. Topological spaces* (X, T) *and* (Y, Υ) *are said to be* **homeomorphic** *iff* $\exists f : X \to Y$ *such that f is a homeomorphism.*

It is often useful to know when two spaces are topologically the same, or homeomorphic, because topologists can then prove properties of one space and apply them to the other. Properties which are preserved under homeomorphism are call *topological properties* or *topological invariants*.

Surfaces

Definition. [Surface]   *A* **surface** *is a Hausdorff space in which every point has a neighborhood about itself homeomorphic to the open unit disc in* $\mathbf{R}^2$.

This means surfaces look like the 2-space locally, which makes a lot of sense intuitively. For example, a sphere is a surface. We can see this because we live on the earth which, if it had its innards removed, would approximate a sphere. We only see the earth on a very local level, and to us it appears to be much like the plane, only stretched a bit here and there.

Homeomorphic Surfaces

Applying the definition of homemorphism to surfaces, we see that two surfaces are homeomorphic iff they can be continuously deformed into one another.

Triangulation

Definition. [Triangulation] *A **triangulation** of a compact surface X consists of a finite collection of closed subsets of X that cover X and a collection of homeomorphisms from the elements of the collection to triangles in the plain. Each member of the collection is also called a **triangle**. We restrict our triangles by stating that the intersection of any two may only contain a single vertex or edge.*

Euler Characteristic

Definition. [Euler characteristic for surfaces] *The* **Euler characteristic** *of a surface* $\chi(X) = v - e + f$, *where* v *represents the number of vertices,* e *represents the number of edges, and* f *represents the number of faces.*

The Euler characteristic is a derivitive of a concjecture of Euler, who supposed that the number of verticies minus the number of edges plus the number of triangles for particular graphs in the plane was always 1.

Euler Characteristic of the Sphere and Torus

See board.

Characteristics of the Surfaces

surface	f	e	v	χ
sphere	1	1	2	2
n tori	1	$2n$	1	$2 - 2n$
m projective planes	1	m	1	$2 - m$

What else do we need for homeomorphism?

Equivilence classes of surfaces have a unique combination of orientabiliy, number of boundary compenents, and Euler charateristic.

Orientability

Definition. [Orientable] *A surface in 3-space is **orientable** if it has two distinct sides, and non-orientable if it has only one side. To put it another way, a surface is non-orientable iff it contains a Mobius band.*

Topological Inequity of Cylinder and Projective Plane

See board.

Bibliography

Kinsey, L. Christine. *Topology of Surfaces.* New York: Springer-Verlag, 1993.

Patty, C. Wayne. *Foundations of Topology.* Boston: PWS-KENT, 1993.

Scott, Jonathan. *Hands on Algebraic Topology.*

http://www.math.toronto.edu/mathnet/simmer/topic.mar98.html

Goldwater, Sharon. *Knots and Surfaces.*

http://www.ai.sri.com/%7egoldwate/math/knots