

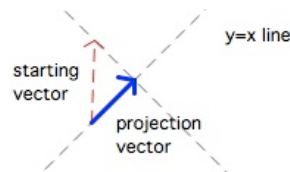
Foxes & Rabbits and Eigenvectors for the Projection Matrix

Recall that the equation $A\mathbf{x} = \lambda\mathbf{x}$ allows us to turn matrix multiplication into scalar multiplication. The eigenvectors are those \mathbf{x} that satisfy the equation. While the 0 vector always works, we look for nontrivial solutions. The eigenvalues λ are the scaling factors - ie we stay on the same line through the origin, and λ tells us how we scale along that line.

- Let's say that foxes and rabbits interact via the following matrix
 $A := \text{Matrix}(\left[\left[\frac{21}{40}, \frac{3}{20}\right], \left[-\frac{3}{16}, \frac{39}{40}\right]\right]);$
 Execute the Eigenvectors(A); command
- Explain why the eigenvectors form a basis for \mathbb{R}^2 .
- Write out the eigenvector decomposition for the system.
- Use the decomposition to explore what will happen to the populations in the longterm, **and** what kind of vector(s) the system will travel along to achieve that longterm behavior, and then fill in the blanks:
 If ___ equals 0 then we die off along the line ___ [corresponding to the eigenvector ___], and otherwise we [choose one: die off or grow or hit and then stayed fixed] along the line ___ [corresponding to the the eigenvector ___].

$$M := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Recall that the matrix M takes (x,y) to $(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x + \frac{1}{2}y)$, ie projected onto the $y=x$ line. If we look at light rays perpendicular to the $y=x$ line, this matrix gives us the shadow a vector makes onto the $y=x$ line, which has many applications in mathematics and physics. In other words, given a starting vector, we drop the perpendicular to the $y=x$ line, and the base of the right triangle we form is the projection vector.

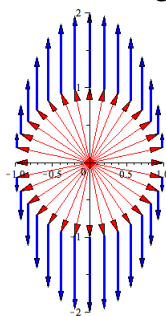


Notice that any vectors on the lines $y=x$ and $y=-x$, with basis representatives $(1,1)$ and $(-1,1)$, are eigenvectors for M . Anything on the line $y=-x$ gets sent to $(0,0)$ which is still on the same line through the origin, so the eigenvalue is 0, and anything on $y=x$ gets fixed by M , so the eigenvalue is 1.

- Input the matrix
 $A := \text{Matrix}(\left[\left[(\cos(\theta))^2, \cos(\theta)\sin(\theta)\right], \left[\cos(\theta)\sin(\theta), (\sin(\theta))^2\right]\right]);$
- Execute Eigenvectors(A); and Eigenvalues(A);

Notice that the eigenvalues are 0 and 1, just like in problem 5.

$\theta = \frac{\pi}{2}$ When θ is $\frac{\pi}{2}$, the line of projection is the y -axis. The following pictures shows the blue outputs in the range (or Image) of A starting at the tip of the inputs so that we can visualize them all on one graph [in actuality all the blue vectors start at the origin].



$\theta = \frac{\pi}{2}$

You should be able to identify the eigenvectors and eigenvalues from the picture.

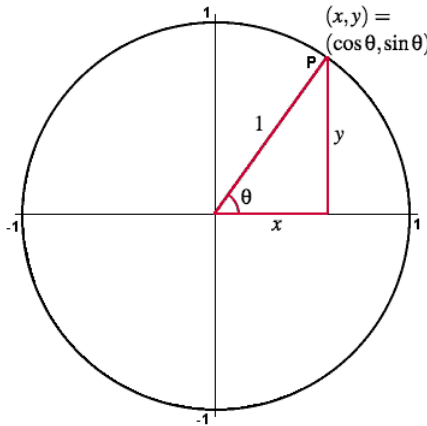
Execute $\theta := \frac{\pi}{2}$; Eigenvectors(A);

and compare with the above picture. The eigenvalue of 1 corresponds to vectors on the line of

projection, and the eigenvalue of 0 corresponds to vectors perpendicular to the line of projection.

The matrix A projects vectors onto the line through the origin that makes an angle of theta degrees with the positive x-axis [in number 1 above, the line was y=x, ie theta was 45 degrees from the positive x-axis. In this case sin and cos are sqrt(2)/2, so the matrix has all entries of 1/2 like M does].

8. Let's use a bit of trigonometry to determine the equation of the line of projection for A for a general theta:



Notice that theta is labeled in the picture above, and I have created a right triangle to the x-axis so that we can use the trigonometry of the unit circle. The hypotenuse is the line of projection for A. Why is the x-value of point P in the picture above cos(theta)? Why is the y-value sin(theta)? Use this to show that the slope of the hypotenuse from (0,0) to P=(cos(theta), sin(theta)) is tan(theta). Since the y-intercept occurs at (0,0), this would tell us that **the line of projection is y=tan(theta) x**.

9. If we use the trigonometry of the unit circle, we can find the eigenvectors for A for any theta. Form P as

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

the matrix of the corresponding eigenvectors as columns

Explain why the first column is an eigenvector corresponding to the eigenvalue of 1, and why the second column is an eigenvector corresponding to an eigenvector of 0. Hint: Where do the vectors lie in relation to the line of projection y=tan(theta) x.

10. What geometric transformation is P? How about MatrixInverse(P)?
11. Notice that if we set up the system Px=0, the only solution would be the trivial solution. Hence the columns of P are linearly independent and so A is diagonalizable. Form
 $A:=\text{Matrix}([(cos(theta))^2,cos(theta)*sin(theta)],[cos(theta)*sin(theta),((sin(theta))^2)]]);$
 $\text{Diag}:=\text{simplify}(\text{MatrixInverse}(P).A.P)$
 What geometric transformation is Diag?
12. Notice that P.Diag.MatrixInverse(P) = A by matrix algebra and # 11. Writing out a transformation in terms of a P, the inverse of P, and a diagonal matrix will prove very useful in computer graphics Recall that we read matrix composition from right to left.

$$P.\text{Diag}.\text{MatrixInverse}(P) = A$$

If we want to project a vector onto the y=tan(theta) x line, first we can perform MatrixInverse(P) which takes a vector and rotates it counterclockwise by theta. Next we perform Diag, which projects onto the line _____. And finally we perform P, which rotates _____ by theta.