First Fundamental Form: Flat and Round Donuts

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. A flat torus or donut cannot be isometrically embedded in $\mathbb{R}^3$ as a smooth C^2 surface without distorting the curvature and geodesics. However, it can be embedded in $\mathbb{R}^4$ by an isometry so that the local intrinsic experience and geometric properties are the same. In the book, Example 5.4.9 beginning on p. 229, John Oprea discusses a parametrization of a flat torus in $\mathbb{R}^4$ as $x(u, v) = [\cos(u), \sin(u), \cos(v), \sin(v)]$. What is a u coordinate curve where we hold v fixed?

2. What is a v coordinate curve?

3. Compute x_u and x_v by-hand and show work. These are the tangent vectors to the respective coordinate curves.

4. By-hand, compute the elements of the first fundamental form

   $E = x_u \cdot x_u$

   $F = x_u \cdot x_v = x_v \cdot x_u$

   $G = x_v \cdot x_v$

5. Discuss what does F tell us here about the coordinate curve tangent vectors at a point?

6. Next write down the metric form ds^2 and compare it with the standard Euclidean metric that gives the Pythagorean theorem.

7. Write the matrix representation of the first fundamental form.

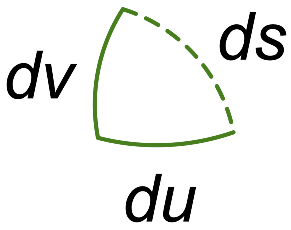
8. Discuss why this surface is called "flat".

   Round Donut in $\mathbb{R}^3$

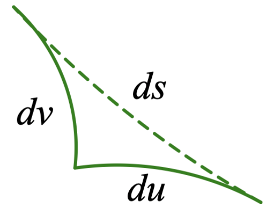
9. First, discuss as you review what happened with the first fundamental form on the sphere and Pythagorean theorem via string on the sphere in the interactive video.

10. For a parametrization of a round donut in $\mathbb{R}^3$, Maple computed $E = 1$, $F = 0$ and $G = (1.1 + \cos(u))^2$, so the matrix representation of the first fundamental form of the round donut is $\begin{bmatrix} 1 & 0 \\ 0 & (1.1 + \cos(u))^2 \end{bmatrix}$. Write down the metric form ds^2.

11. Given the physical round donut and string in front of you, which would have a different parametrization and different first fundamental form, investigate the Euclidean Pythagorean theorem on the outside of the donut by using the string. Is the torus hypotenuse on the outside of the donut too short when compared to a flat triangle hypotenuse, like happens on the sphere, or too long, which happens in hyperbolic space and saddle surfaces. See the images, metric forms, and length statements below and circle one set:



OR



$du^2 + dv^2 = ds^2_{flat} > ds^2_{outsidedonut}$ OR $du^2 + dv^2 = ds^2_{flat} < ds^2_{outsidedonut}$

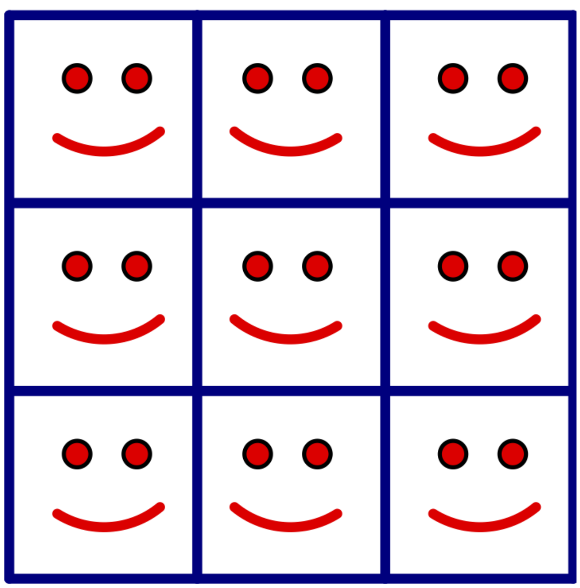
torus hypotenuse on the outside is too short OR torus hypotenuse on the outside is too long

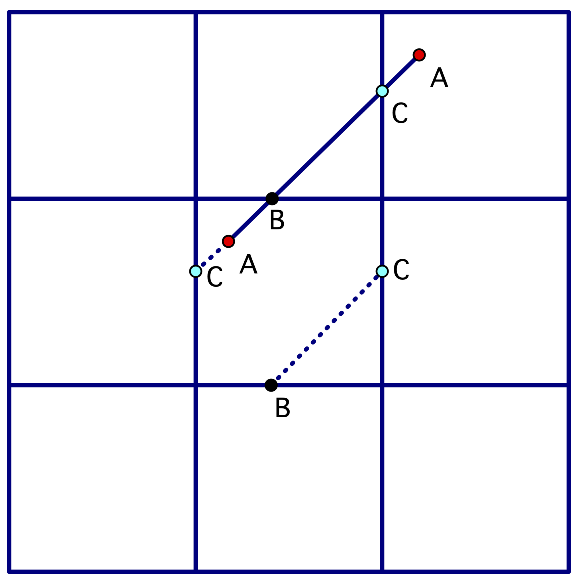
12. How about the inside of the physical donut? Is the torus hypotenuse too short or too long? Also draw a corresponding image and write a corresponding metric form equation.

Back to the Flat Torus in $\mathbb{R}^4$—Geodesics

13. A flat torus can be theoretically obtained by taking a square and identifying the edges straight across (top to bottom and separately left to right) without distorting the geometry of the interior. So a covering space would be infinitely squares next to each other which are exact copies of each other. How many geodesics typically join two points in a flat torus?

14. On the left is a picture of a few of the squares in the covering—-they would continue on in all directions. On the right is a straight line geodesic in the covering that begins at A in the center square and returns to the A that is 1 square up and another square over, making a loop as the geodesic closes back up on itself. We can consider the path of the geodesic completely in the center square by continuing at identified points, like I did with the dotted lines. Make sense of this closed geodesic and discuss with your group.





15. Can a geodesic on a flat torus ever intersect itself in angles that differ from integer multiples of π? If so give an example through pictures you draw in the covering space and if not, explain why not. Discuss and then respond on `pollev.com/drsarah314`

a) yes and I have a good reason why

b) yes but I am unsure of why

c) no but I am unsure of why not

d) no and I have a good reason why not