Christoffel Symbols

Christoffel symbols and geodesics are important in relativity because objects travel along geodesic paths in spacetime, and geodesics are expressed in terms of Christoffel symbols. Once we have a metric form

$$ds^2 = g_{ab}dx^a dx^b$$

the Christoffel symbols are given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}).$$

Here g^{ab} is the corresponding element in the matrix inverse of the first fundamental form matrix g_{ab}. (The same symbol is often used interchangeably to represent the matrix and the entry). The Christoffel symbols are intrinsic quantities and they tell us how to take covariant derivatives on the manifold. These are coefficients of tangent vectors (more generally connection coefficients), in the same way we showed they arose for surfaces. Just like there, the geodesic equation sets the tangential components of the second derivative equal to 0 so that vectors that start parallel stay so via parallel transport—i.e. that the bug doesn't feel any intrinsic curvature.

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

These are the differential equations for a geodesic expressed in local coordinates. This has theoretical importance in mathematics and physics in analytic treatments of geodesics, but in practice, these equations can rarely be solved, except approximately. The easier way to compute both the geodesic equations and the Christoffel symbols at once is as follows. Let the Lagrangian

$$I = g_{ab}\dot{x}^a\dot{x}^b.$$

Then a geodesic will satisfy the Euler-Lagrange equations

$$\frac{d}{ds}(\frac{\partial I}{\partial \dot{x}^a}) - \frac{\partial I}{\partial x^a} = 0 \text{ for all } a.$$

Once we calculate an equation for each a, we can compare to the geodesic equation in order to read off the Γ^a_{bc} Christoffel symbols, because both the Euler-Lagrange equation and the geodesic equation will be expressed in terms of $\ddot{x}^a$.

1. Consider the metric

$$ds^2 = dx^2 + dy^2 = dxdx + 0dxdy + 0dydx + dydy$$

   (a) What is x^1 for $a = 1$? What is x^2? What is the matrix g_{ab}?

(b) Let $I = g_{ab}\dot{x}^a\dot{x}^b = \dot{x}^2 + \dot{y}^2$. Find all of the Christoffel symbols using the Euler-Lagrange equations as follows:

When $a = 1$ in Euler-Lagrange Equation is:

$0 = \frac{d}{ds}(\frac{\partial I}{\partial \dot{x}^1})$ - $\frac{\partial I}{\partial x^1}$

We can write this in terms of $x^1 = x$:

$0 = \frac{d}{ds}(\frac{\partial I}{\partial \dot{x}}) - \frac{\partial I}{\partial x}$

Next take the relevant partials and derivatives: $0 = \frac{d}{ds}(2\dot{x}) - 0 = 2\ddot{x}$

Therefore, the geodesic equation given by: $\ddot{x} + \Gamma^1_{bc}\,\dot{x}^b\,\dot{x}^c = 0$ allows us to read off the Christoffel Symbols as we compare the expanded geodesic equation (where $x^1 = x$):

(1) $\quad 0 = \ddot{x}^1 + \Gamma^1_{11}\,\dot{x}^1\dot{x}^1 + \Gamma^1_{12}\dot{x}^1\dot{x}^2 + \Gamma^1_{21}\dot{x}^2\dot{x}^1 + \Gamma^1_{22}\dot{x}^2\dot{x}^2$

to what we calculated above from the Euler-Lagrange Equation: $0 = 2\ddot{x}$, i.e.

(2) $\quad 0 = \ddot{x} = \ddot{x}^1$

So compare (1) to (2) to read off the Christoffel symbols that have $a = 1$. What are each of the four Γ^1_{ab} Christoffel symbols?

(c) Now let $a = 2$ in the Euler-Lagrange Equation $\frac{d}{ds}(\frac{\partial I}{\partial \dot{x}^a}) - \frac{\partial I}{\partial x^a} = 0$ for all a. To get used to Einstein summation notation, write out the generic Euler-Lagrange equation with $a = 2$.

(d) Next write the equation in terms of $x^2 = y$ and take the relevant partials and derivatives.

(e) Then write the expansion of the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ using $a = 2$.

(f) Compare to find the four Γ^2_{ab} Christoffel symbols?

We can also use the Euler-Lagrange equations to show that the geodesics are straight lines as each coordinate is linear with respect to s.

$0 = 2\ddot{x} \implies x = c_1 s + c_2$

$0 = 2\ddot{y} \implies y = c_3 s + c_4$

2. Consider the sphere of radius r and the metric form $ds^2 = r^2 d\phi^2 + r^2 sin^2(\phi) d\theta^2$

   (a) What is x^1 for $a = 1$? What is x^2? What is the matrix g_{ab}?

   (b) Write $I = g_{ab}\dot{x}^a\dot{x}^b$.

   (c) Let $a = 1$ in the Euler-Lagrange Equation $\frac{d}{ds}(\frac{\partial I}{\partial \dot{x}^a}) - \frac{\partial I}{\partial x^a} = 0$ for all a. To get used to Einstein summation notation, write out the Euler-Lagrange equation with the $a = 1$ angle.

   (d) Next take the relevant partials and derivatives.

   (e) Then write the expansion of the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ using $a = 1$.

   (f) Compare to find the four Γ^1_{ab} Christoffel symbols?

   (g) Repeat to find the four Γ^2_{ab} Christoffel symbols?

We could solve the geodesic equations to show that they give us great circles. In general, once we have the Christoffel symbols, we can define all sorts of useful intrinsic equations and tensors.
Riemann curvature tensor or Riemann-Christoffel tensor $R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$
Ricci tensor $R_{ab} = R^c_{acb} = g^{cd}R_{dacb}$
Scalar curvature $R = g^{ab}R_{ab}$
Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$