Geodesic Worksheet.

Adapted from Amy Ksir's Relativity Course at the Naval Academy

Geodesics are important in relativity because objects travel along geodesic paths in spacetime. We will describe a path on our *n*-dimensional manifold by . Once we have a metric

Metric: 

on the manifold, the Christoffel symbols are given by

Christoffel symbols: 

Where  is the inverse of the first fundamental form matrix . Christoffel symbols are intrinsic quantities and an intrinsic definition of a geodesic will be a path satisfying that the geodesic curvature is 0:

Geodesic: 

These are the differential equations for a geodesic expressed in local coordinates. This has theoretical importance in mathematics and physics in analytic treatments of geodesics, but in practice, these equations can rarely be solved, except approximately.

The easy way to compute both the geodesic equations and the Christoffel symbols at once is as follows: Let . Then a geodesic will satisfy the

Euler-Lagrange equations: 

for all *a*. It will also satisfy *I* = *k* for some constant *k*.

The Christoffel symbols tell us how to take covariant derivatives on the manifold (not a tensor, but can be used to form tensors).

1. Consider the metric  (flat 2D space in Cartesian coordinates).

a) What is ? What is *I*?

b) Let . Find all of the Christoffel symbols using the Euler-Lagrange equations.

c) Using the Euler-Lagrange equations, show that the geodesics are straight lines.

2. Consider the sphere of radius *a*. Using the equations

, , ,

a) Find *dx*, *dy*, and *dz*.

b) Show that the flat metric on 3D space induces the metric 

c) Using the Euler-Lagrange equations, find the Christoffel symbols and equations for a geodesic.

Our book solves these on p. 216-218 using a different parametrization. Consider the plane  through the origin. The curve generated by intersecting the plane with the sphere (a great circle) satisfies the geodesic equations. Our earlier proof that geodesics must be great circles wrote the surface normal U = γ/R, which was used in γ’’ along with Frenet equation arguments.

We can define all sorts of useful intrinsic equations and tensors once we have the Christoffel symbols: 

Geodesic: 

Curvature tensor: 

Ricci tensor: .

Einstein tensor: .

For a two-dimensional surface, the principal curvatures  and  are the most upward (positive) and the most downward (negative) curvatures. They measure the amount that the surface is curving in space and so they cannot be measured by a bug confined to the surface. Gauss made the astonishing discovery, however, that the Gauss curvature  is intrinsic. This generalizes to higher dimensions via the curvature tensor.