Additional Graduate Problems for Project 1

Graduate students will complete the undergraduate assignment and, in addition, turn in your responses to the following, so you will have 3 problems turned in, with some choice for the first two. See the instructions in the undergraduate assignment, which apply here too.

1. Select one of the following:

   (a) **Graduate Problem: Computer Images of Curves**

   - In Maple, use a plot command to plot the planar curve $y = \sqrt{10^{-30} + x^2}$. When plotted from x=-1..1, the curve looks like it behaves the same as $y = \|x\|$ at the origin. Test out smaller ranges of x in the form of $\frac{1}{10...0}$(i.e. $x = -\frac{1}{10}..\frac{1}{10}$, etc). Can you distinguish the behavior of these curves at the origin by a similar Maple plot command? If so, what is the largest value of x of the form $x = -\frac{1}{100...0}..\frac{1}{100...0}$ that can distinguish them?
   - Can you distinguish the behavior of the curves at the origin using differential geometry techniques? If so, explain how, and show work to distinguish the curves at the origin.

   OR

   (b) **Graduate Problem: Curve Proof for an Osculating Plane**
   Let α be a curve with $\kappa > 0$. Show that if α's osculating planes have a point in common, call it p (i.e. each plane passes through p), then α is planar.

2. Select one of the following:

   (a) **Graduate Problem: Elliptic Integrals**

   For some integrals, like arc length, Maple outputs an elliptic function rather than a closed form. Research elliptic integrals and summarize what you have learned, being sure to include any references back in #15 of the undergraduate assignment.

   OR

   (b) **Graduate Problem: The Darboux Vector $\omega(s)$ via Cross Products**
   We used dot product arguments to prove examine $\omega(s)$, the Darboux vector. Complete Exercise 1.3.12 on p. 21, which directs use of a cross product argument. You may assume and use the relationships $T \times N = B$, $N \times B = T$, and $B \times T = N$.

3. **Graduate Problem: Curve Proofs for the Fundamental Theorem of Space Curves**

   - Read a proof typeset by Robert Hardt (see below), which uses vector valued functions and ODEs to prove the Fundamental Theorem of Space Curves. Note that the wedge symbol is used for cross product and Hardt uses the negation of our torsion for their torsion in the last line at the bottom of page 1. For that very last like, redo with ours and fill in the argument for just the last line at the bottom of Hardt's page 1. You may assume that cross product relationships hold: $T \times N = B$, $N \times B = T$, and $B \times T = N$ and that the Frenet-Serret equations hold for the TNB derivatives.
   - Next, complete exercise 1.5.17 on p. 38 to sketch a proof of the Fundamental Theorem of Space Curves using transformations of 3-space. We will work on this in class, so you may want to wait until then to complete this portion.
   - Compare and contrast the arguments.

Handout 3. Fundamental Theorem of Curves

Theorem. *Given a smooth function $\tau(s)$ and a positive smooth function $\kappa(s)$ on an interval I containing 0, a point $\alpha_0 \in \mathbf{R}^3$, and two unit vectors T_0 and N_0 in $\mathbf{R}^3$, there exists a unique unit-speed curve $\alpha(s)$ on I with curvature $\kappa(s)$, torsion $\tau(s)$, initial position, $\alpha(0) = \alpha_0$, initial velocity $\alpha'(s) = T_0$, and initial acceleration $\alpha''(s) = \kappa(0)N_0$.*

Proof. For the vector-valued function $u \equiv (T, N, B) : I \to \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$, we may solve the O.D.E.

$$\begin{aligned} T' &= \kappa N\ , \\ N' &= -\kappa T \ + \ \tau B\ , \\ B' &= -\tau N \end{aligned}$$

with initial conditions $T(0) = T_0,\ N(0) = N_0,\ B(0) = T_0 \wedge N_0$.

We claim that (T, N, B) is then automatically an orthonormal frame. To see this, note that vector-valued function

$$v = (v_1, v_2, v_3, v_4, v_5, v_6) \equiv (T \cdot T, T \cdot N, T \cdot B, N \cdot N, N \cdot B, B \cdot B)$$

then satisfies the O.D.E.

$$\begin{aligned} v_1' &= -2\kappa v_2\ , \\ v_2' &= -\kappa v_4 \ - \ \kappa v_1 \ + \ \tau v_3\ , \\ v_3' &= \kappa v_5 \ - \ \tau v_3\ , \\ v_4' &= -2\kappa v_2 \ + \ 2\tau v_6 \\ v_5' &= -\kappa v_3 \ + \ \tau v_6 \ - \tau v_4\ , \\ v_6' &= -2\tau v_5 \end{aligned}$$

with $v(0) = (1, 0, 0, 1, 0, 1)$. But since the *constant* function $(1, 0, 0, 1, 0, 1)$ also satisfies the above O.D.E. with the same initial data, we conclude that $v \equiv (1, 0, 0, 1, 0, 1)$, so that (T, N, B) is indeed an orthonormal frame.

Also since the length of the vector function $u = (T, N, B)$ remains bounded, in fact identically $\sqrt{3}$, we see, by continuation, that the solution (T, N, B) exists not only near $s = 0$ but even over the whole interval I.

We conclude that

$$\alpha(s) \ = \ \alpha_0 \ + \ \int_0^s T(t)\, dt$$

has $\alpha(0) = \alpha_0$ and is unit-speed with tangent T because $\alpha'(s) = T(s)$. Also

$$\alpha''(s) \ = \ T'(s) \ = \ \kappa N(s)$$

so that α has curvature κ and principal normal N. Moreover,

$$(T \wedge N)'(s) \ = \ (T' \wedge N)(s) \ + \ (T \wedge N')(s) \ = \ \tau(s) N(s)$$

so that α also has torsion τ.

Finally, if $\overline{\alpha}$ is another unit-speed curve with curvature κ and torsion τ, $\overline{\alpha}(0) = \alpha_0$, $\overline{\alpha}'(0) = T_0$, and $\overline{\alpha}''(0) = \kappa(0)N_0$, then the function $f(s) = T \cdot \overline{T} + N \cdot \overline{N} + B \cdot \overline{B}$ satisfies $f(0) = 3$ and, by the Frenet formulas, $f' \equiv 0$. Thus f is the constant function 3, and each of the at most unit-sized terms $T \cdot \overline{T},\ N \cdot \overline{N},\ B \cdot \overline{B}$ must be the constant 1. Being unit vectors, $T \equiv \overline{T},\ N \equiv \overline{N},\ B \equiv \overline{B}$. In particular,

$$\overline{\alpha}(s) \;=\; \alpha_0 \;+\; \int_0^s \overline{T}(t)\,dt \;=\; \alpha_0 \;+\; \int_0^s T(t)\,dt \;=\; \alpha(s)\ .$$

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