SpaceTime-Time

http://www.spacetime-model.com/img/mass/einstein.jpg

4D Manifold, g_{ij}, curvature satisfy Einstein field equation
g_{ij} can be other 4x4 symmetric matrices acting as $w^T g_{ij} v$.
g_{ij} 1 eigenvalue > 0 and three eigenvalues < 0 at each point.
angle : $\cos\theta = \frac{w^T g_{ij} v}{|v||w|}$, spacetime interval: $|v| = \sqrt{v^T g_{ij} v}$

- change g_{ij} and change physical and geometric properties of spacetime

Minkowski vs Wormhole Metric

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} ds^2 = g_{ab}dx^a dx^b \; ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

where t is time and x, y, z are rectangular coordinates in space

$$g_{ij} = \begin{bmatrix} -1 + \frac{r_0}{r} - \frac{\epsilon}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_0}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

$$ds^2 = (-1 + \frac{r_0}{r} - \frac{\epsilon}{r^2})dt^2 + \frac{1}{1 - \frac{r_0}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where t is time and r is radial, θ, ϕ are angular, ϵ is electric charge, r_0 is smallest radius of throat

Curvatures

metric form

$$ds^2 = g_{ab} dx^a dx^b$$

Christoffel symbols

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}).$$

geodesics

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0$$

Riemann curvature tensor or Riemann-Christoffel tensor
$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed}$
Ricci tensor $R_{ab} = R^c_{acb} = g^{cd} R_{dacb}$
Scalar curvature $R = g^{ab} R_{ab}$
Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

Christoffel Symbols and Curvatures

The Christoffel symbols

- intrinsic quantities, how to take covariant derivatives
- coefficients of tangent vectors (connection coefficients)
- measure whether or not vectors are parallel transports
- in relativity, gravitational forces. geodesics and curvatures.

Riemann curvature tensor: measures how much a manifold is not flat via $4^4 = 256$ entries for spacetime.

Ricci tensor: trace (sum of diagonal elements) relates to the metric volume $\sqrt{det g_{ij}}$.

Scalar curvature: number. For surfaces—twice Gaussian curvature. For relativity—Lagrangian density.

Einstein tensor describes curvature of spacetime due to the presence of energy or mass, has zero divergence

Maple file on the wormhole

The metric:

```
> g_components:=array(symmetric,1..4,1..4,[[-(1-r0/r+epsilon/r^2),0,0,0],[0, 1/(1-r0/r),0,0],[0,0,r^2,0],[0,0,0,r^2*sin(theta)^2]]);
```

$$g\_components := \begin{bmatrix} -1+\frac{r0}{r}-\frac{\epsilon}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r0}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix} \quad (1)$$

▶ Details of the Computations

Christoffel Symbols Gamma_ab^c

```
> Christoffel2(ginv,Cfl1);
```

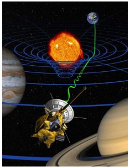
$$\text{table}\Big(\Big[compts = \text{array}\Big(cf2, 1..4, 1..4, 1..4, \Big[(1,1,1)=0, (1,1,2)=\frac{(-r^2+r0\,r+\epsilon)(-r0\,r+2\epsilon)}{2(r-r0)^2 r^3}, (1,1,3)=0, (1,1,4)=0, (1,2,1)=\frac{(-r^2+r0\,r+\epsilon)(-r0\,r+2\epsilon)}{2(r-r0)^2 r^3}, (1,2,2)=0, (1,2,3)=0, (1,2,4)=0, (1,3,1)=0, (1,3,2)=0, (1,3,3)=0, (1,3,4)=0, (1,4,1)=0, (1,4,2)=0, (1,4,3)=0, (1,4,4)=0, (2,1,1)=-\frac{(r-r0)(-r0\,r+2\epsilon)}{2r^4}, (2,1,2)=0, (2,1,3)=0, (2,1,4)=0, (2,2,1)=0, (2,2,2)=-\frac{r0}{2(r-r0)r}, (2,2,3)=0, (2,2,4)=0, (2,3,1)=0, (2,3,2)=0, (2,3,3)=-r+r0, (2,3,4)=0, (2,4,1)=0, (2,4,2)=0, (2,4,3)=0, (2,4,4)=-(r-r0)\sin(\theta)^2, (3,1,1)=0, (3,1,2)=0, (3,1,3)=0, (3,1,4)=0, (3,2,1)=0, (3,2,2)=0, (3,2,3)=\frac{1}{r}, (3,2,4)=0, (3,3,1)=0, (3,3,2)=\frac{1}{r}, (3,3,3)=0, (3,3,4)=0, (3,4,1)=0, (3,4,2)=0, (3,4,3)=0, (3,4,4)=-\sin(\theta)\cos(\theta), (4,1,1)=0, (4,1,2)=0, (4,1,3)=0, (4,1,4)=0, (4,2,1)=0, (4,2,2)=0, (4,2,3)=0, (4,2,4)=\frac{1}{r}, (4,3,1)=0, (4,3,2)=0, (4,3,3)=0, (4,3,4)=\frac{\cos(\theta)}{\sin(\theta)}, (4,4,1)=0, (4,4,2)=\frac{1}{r}, (4,4,3)=\frac{\cos(\theta)}{\sin(\theta)}, (4,4,4)=0\Big]\Big), index\_char=[1,-1,-1]\Big]\Big)$$

Riemann Curvature Tensor

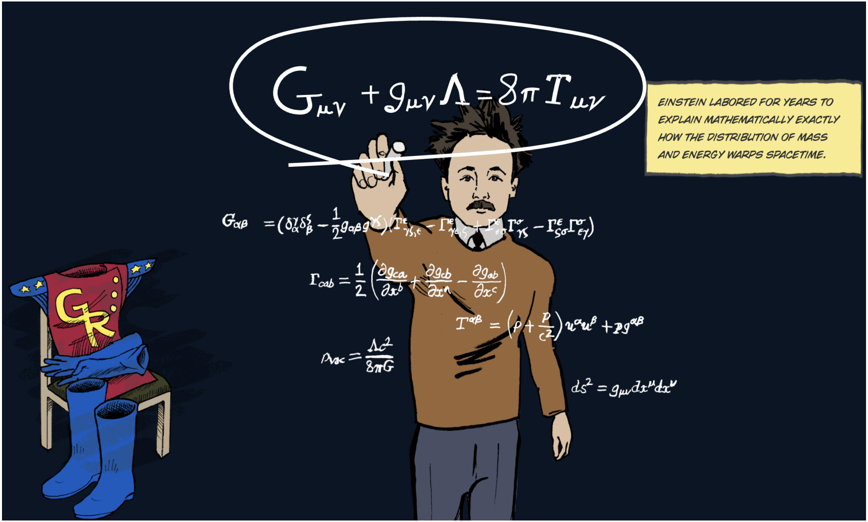
```
> Rmn:=Riemann(ginv, D2g, Cfl1);
```

General Relativity

- Precession of the orbit of Mercury: Newton used solar gravitational attraction and calculus to explain Kepler's elliptical planetary orbits but the orbit rotated (precessed) at an unexpected rate
- Spacetime in the presence of masses is curved and geodesics more interesting
- Gravity is the curvature of spacetime
- Arthur Eddington (1919): star near sun shifted by amount predicted by relativity! $\rightarrow$ Einstein public figure
- Radio sources
- Gravitational lensing and LIGO gravitational waves



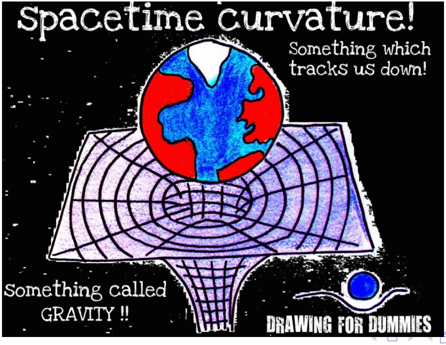
https://science.nasa.gov/science-pink/s3fs-public/mnt/medialibrary/2004/03/24/



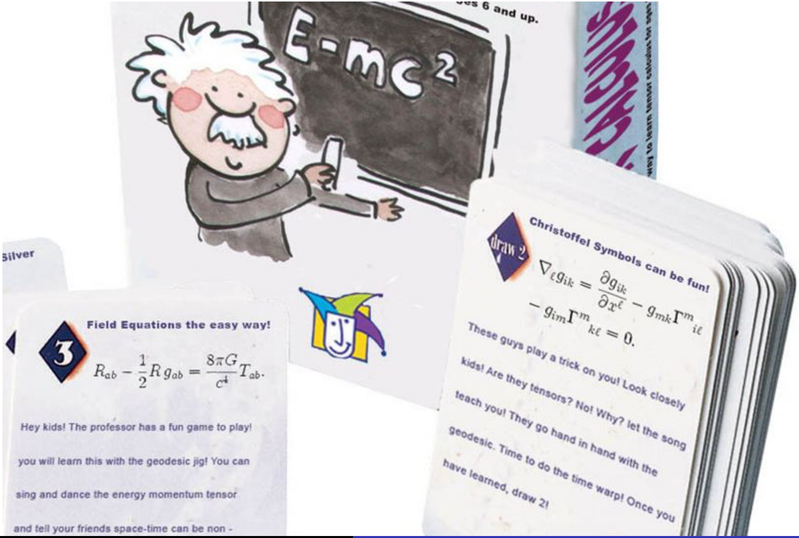
https://vis.sciencemag.org/generalrelativity/

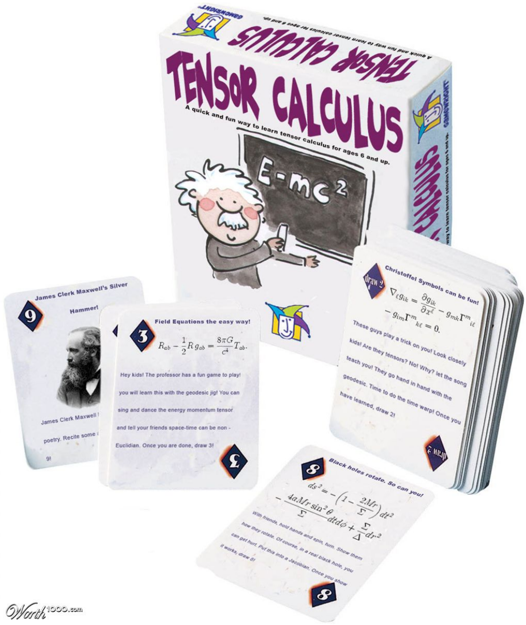
Imagine That

- *In the period between the publication of special relativity (1905) and general relativity... he took some time to learn enough differential geometry to develop his ideas. This apparently did not come easily to him, and involved a lot of consultation with other people in Europe. He and Levi-Civita were in very frequent communication for example.*



- Einstein's general relativity remains scientists' best understanding of gravity and a key to our understanding of the cosmos on the grandest scale





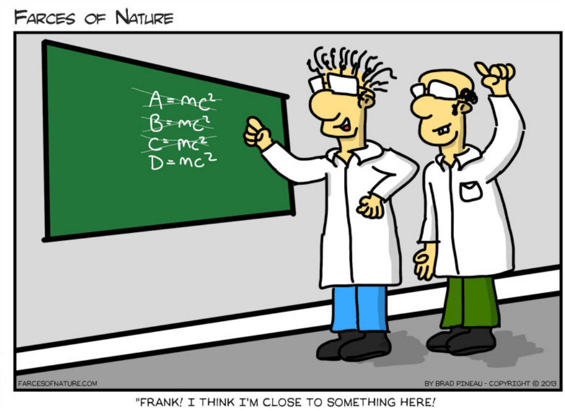
http://worth1000.s3.amazonaws.com/submissions/309500/309571_1db4_1024x2000.jpg

January 7, 1943

Dear Barbara:

I was very pleased with your kind letter. Until now I never dreamed to be something like a hero. But since you have given me the nomination, I feel that I am one...

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.



You may work alone or in a group of up to 2 people and turn in one per group.
Metric forms will be assigned on a **first come-first-served** basis in the choice selection feature on ASULearn. If you are working in a group, one person selects the metric form and the other selects the option "working with someone else who already selected our topic."

- Alcubierre metric or warp drive metric
- anti-de Sitter metric
- de Sitter metric for special relativity
- Eddington-Finkelstein coordinates
- Friedmann-Lemaître-Robertson-Walker (FLRW) metric
- Gödel metric
- Gullstrand-Painlevé coordinates
- Kerr metric
- Kerr-Newman rotating charged black hole metric
- Kruskal-Szekeres coordinates
- Lemaître coordinates
- Minkowski metric/space
- Reissner-Nordström metric
- Rindler coordinates
- Schwarzschild metric
- Taub-NUT metric
- Weyl-Lewis-Papapetrou coordinates
- Wormhole metric
- Other interesting metric forms may be approved

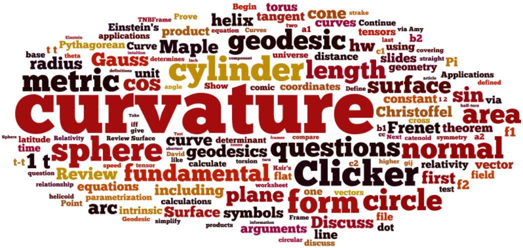
Research

Explore the following via researching and (keep track of ALL your references for #7).

1. Write down a metric form for your topic (like $ds^2 = ...$) and summarize what any variables stand for.

Course Overview

- Curves: torsion and curvature, piece together (DEs) to give us a nice curve. Frenet frame basis
- Surfaces: 1st and 2nd fundamental forms. 1st and Christoffel symbols intrinsic $\to$ Gauss curvature. E, F, G piece together to give us a nice smooth surface, or one with manageable singularities like the cone. l, m, n determine how the surface sits in space (if it does)
- SpaceTimes: g_{ij} and Γ_{ij}^k more terms $\to$ curvature tensors. Einstein's field equations: way to solve for "nice" solutions.



- Final Research Presentation: extension + bib + peer review