DIAMETERS OF SPHERICAL ALEXANDROV SPACES AND CURVATURE ONE ORBIFOLDS

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Abstract. Let G be a closed, non-transitive subgroup of $O(n+1)$, where $n \geq 2$, and let $Q^n = S^n/G$. We will show that for each n there is a lower bound for the diameter of Q^n. If G is finite then Q^n is an orbifold of constant curvature one and an explicit lower bound can be given. For Coxeter groups, the resulting lower bound is independent of dimension. Otherwise, Q^n is a spherical Alexandrov space and we will show existence of a lower bound. In the process, we will compute some examples of quotient spaces and their diameters.

1. Introduction

While representations of compact Lie groups are well understood, the geometry of the corresponding spherical quotients is virtually unknown. Let G be a closed, non-transitive subgroup of $O(n+1)$, where $n \geq 2$, and let $Q^n = S^n/G$. The goal of this paper is to find lower bounds for the diameter of Q^n.

Knowing how small the diameter can get not only gives information about Q^n itself, but can also lead to other interesting results. For example, for the equivariant sphere theorem in [21], let G be a closed subgroup of the isometries of a closed manifold M with positive sectional curvature. Given any point p on the manifold, the tangent space to the orbit Gp at p is invariant under the isotropy group at p, which is a subgroup of the orthogonal group, $O(T_pM)$. Hence, the normal space is also left invariant. Let $S_{[p]}$ be the quotient of the unit sphere in this normal space by the isotropy group at p. This is one of the above spaces Q^n. If two points p and q can be found on M so that the diameters of $S_{[p]}$ and $S_{[q]}$ are both less than $\frac{\pi}{4}$, then M is the union of tubular neighborhoods about the orbits of these points. Thus, local diameter information gives global results about the structure of the manifold.

For lower bounds on the diameter, it is only necessary to examine irreducible actions since the diameter of a reducible action is π when there is a point of S^n fixed by the entire group, and $\frac{\pi}{2}$ otherwise (see [21] and [4]).

If the group is finite, then the resulting quotient space is a constant curvature one orbifold. If Γ acts properly discontinuously and freely, then S^n/Γ is a manifold. All such groups are classified in [34]. McGowan [25] used this classification to show that diameters

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of these manifolds are bounded below by $\frac{1}{2}\arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is approximately $\frac{\pi}{9.63}$ (see also [18]). This lower bound is optimal and occurs in dimension $n = 3$.

The problem of classifying all finite subgroups of $O(n+1)$ is equivalent to classifying all orthogonal representations of finite groups. Hence, methods of exhaustive computation, as in the manifold case, are not feasible beyond a few small dimensions.

After providing the necessary background in Section 2, this paper discusses lower bounds on the diameter of S^n/Γ when Γ is closed and non-transitive, applications, and an exhaustive computation of the resulting spaces and their diameters for a wide class of groups.

We first examine the diameters of spaces resulting from finite groups in Section 3.

Theorem 3.14 *If Γ is finite, then there exists ϵ, depending only on n, so that* $\mathsf{diam}(\mathsf{S^n}/\Gamma) \geq \epsilon(\mathsf{n})$.

The lower bound $\epsilon(n)$ is explicit in the proof of the theorem, although it does tend to 0 as n gets large. A brief sketch of the proof follows. Using ideas in the proof of Bieberbach's first theorem [9], it can be shown that elements within a neighborhood of the identity commute, since Γ is a finite group. Then, a finite index abelian subgroup of Γ can be found so that the index can be universally controlled. The diameter of the abelian subgroup, which we prove to be at least $\frac{\pi}{2}$, and the index are used to bound the diameter of S^n/Γ by a constant which depends only on n.

Corollary 3.14.1 *If Γ is a finite subgroup of the isometry group of M^n, where M^n is $\mathbb{C}P^n$ or $\mathbb{H}P^n$, then there exists ϵ depending only on n, so that* $\mathsf{diam}(\mathsf{M^n}/\Gamma) \geq \epsilon(\mathsf{n})$.

The proof is an extension of Theorem 3.14 via a Hopf-fibration argument.

Theorem 3.15 *If Γ is a Coxeter group, generated by reflections, then* $\mathsf{diam}(\mathsf{S^n}/\Gamma) \geq \frac{\pi}{8.10}$.

This is achieved as the diameter of a quotient of S^3. The diameter of Coxeter orbifolds, resulting from groups generated by reflections, increases monotonically in n, and as n approaches infinity, the diameter approaches $\frac{\pi}{2}$.

We next examine the diameters of spaces resulting from closed, infinite, nontransitive groups in Section 4. If the group is infinite, then the resulting quotient space is an Alexandrov space with curvature bounded below. The explicit orbifold lower bounds in Theorem A do not apply since discreteness was needed in the proof. When the quotient space is an interval, the action is called a cohomogeneity-one action. Since the orbits Γp are isoparametric hypersurfaces in spheres, the length of the intervals are at least $\frac{\pi}{6}$ (see [23]). There are only two examples, in dimensions 7 and 13, where the diameter is equal to $\frac{\pi}{6}$. However, unlike the manifold and Coxeter orbifold cases, there is an entire class of cohomogeneity-one actions on S^n, including actions for arbitrarily large dimensions, which result in a quotient space of diameter $\frac{\pi}{4}$.

Theorem 4.3 *If Γ is a non-transitive group, then there exists ϵ, depending only on n, so that* $\mathsf{diam}(\mathsf{S^n}/\Gamma) \geq \epsilon(\mathsf{n})$.

The lower bounds are not explicit since the proof is by contradiction. We show that a non-transitive sequence of groups cannot converge to a transitive subgroup of $SO(n+1)$. The proof relies on a theorem of Montgomery and Zippin [28], which says that groups converging to a Lie group must eventually be conjugate to subgroups. Corollary 3.14.1 is needed when the transitive group is not simple.

Finally, we summarize the diameter results in Table 2. For quotient space orbifolds, we find lower bounds which decrease to 0 as n gets large, but it is an interesting question whether optimal lower bounds actually do decrease with n, or whether the optimal lower bound is always achieved in dimension 3, as in the manifold and Coxeter orbifold cases, and whether it is always true, as in the manifold and Coxeter orbifold cases, that the optimal lower bound for the diameter increases to $\frac{\pi}{2}$ as n goes to infinity.

I would like to thank my advisor, Wolfgang Ziller, for his guidance and patience. I am grateful to Karsten Grove, whose idea of taking limits of groups was the beginning of the proof of Theorem 4.3, and Kevin Whyte, for the idea of the proof of Theorem 3.14.

2. Background

2.1. **Riemannian Orbifolds.** We present some basic ideas and intuition about Riemannian orbifolds along with references where rigorous definitions and proofs can be found.

While a manifold locally looks like Euclidean space, $\mathbb{R}^n$, [13], an orbifold locally looks like the quotient of $\mathbb{R}^n$ by a discrete group action (see [29, page 662] or [30, section 2]).

A Riemannian orbifold locally looks (isometrically) like the quotient of a Riemannian manifold by a finite subgroup of its isometry group [4, pages 9–12]. See [4, pages 24–28] and [20, chapter 3] for examples of Riemannian orbifolds.

Remark 2.1. In general, an orbifold is not even homeomorphic to a manifold. For example, look at $x \to -x$ as a $\mathbb{Z}_2$ action on $\mathbb{R}^3$. Now, $\mathbb{R}^3/\mathbb{Z}_2$ is homeomorphic to a cone on $\mathbb{R}P^2$, but is not homeomorphic to a manifold at the cone point. However, in dimension 2, any orbifold is homeomorphic to a manifold [30, page 422]. Yet, orbifolds with cone points are not isomorphic or isometric to manifolds.

Remark 2.2. For the purpose of this paper, we drop the Riemannian label and assume that all of our orbifolds are Riemannian.

One can measure distance locally on an orbifold via lifting up to the Riemannian manifold to compute lengths. To measure distance globally, we add up local lengths. While these local lifts are not unique, the length of a curve is well defined [4, pages 18–22]. One can define orbifold curvature as the curvature of the Riemannian manifold in the local lift. Other Riemannian geometric concepts can also be extended to orbifolds. For example, Toponogov's Theorem, Volume Comparison, Sphere Theorems, Finiteness Theorems and The Closed Geodesic Problem are all discussed in [4].

A good orbifold is the global quotient of a Riemannian manifold by a discrete subgroup of its isometry group [4, page 11]. A bad orbifold is an orbifold which does not arise in this manner.

Lemma 2.3. *([32]) There are no bad constant curvature orbifolds*

Idea of Proof of Lemma 2.3 One can construct a developing map into the constant curvature space form $M = S^n, \mathbb{R}^n$, or $\mathbb{H}^n$ from the orbifold universal covering using the fact that any of the local charts that are used to define constant curvature orbifolds are restrictions of global actions. (See [29, chapter 13] for definitions of the developing map and orbifold universal cover).

Remark 2.4. Let $\Gamma \subset O(n+1)$ be finite. Then $O^n = S^n/\Gamma$ is a constant curvature one orbifold.

2.2. **Spherical Alexandrov Spaces.** If G is not a discrete group, but is instead a closed infinite group, then S^n/G is a spherical Alexandrov space with curvature bounded below. This is a length space with Riemannian notions such as distance and curvature are obtained by comparison with S^n via Toponogov [8, 31].

2.3. **Reducible Orthogonal Actions and the Diameter.**

Lemma 2.5. *([5],[26]) Let $O^n = S^n/G$, where $G \subset O(n+1)$ is closed. Then, $\mathsf{diam(O)} > \pi/2$ iff $\mathsf{diam(O)} = \pi$ iff there is a point of S^n fixed by the whole group.*

Lemma 2.6. *([5],[26]) Let $O^n = S^n/G$, where $G \subset O(n+1)$ is closed. Then, G is a reducible action iff $\mathsf{diam(O)} = \frac{\pi}{2}$ or π*

Reducible actions lead to a large diameter of at least $\frac{\pi}{2}$ by the above lemmas. In addition, the resulting diameter corresponding to a group is no larger than the resulting diameter corresponding to its subgroups [26].

3. Finite Groups

3.1. **Intuition and Examples.** As we saw in Section 2.3, it is only necessary to examine irreducible actions for a lower bound on the diameter, since the diameter is π when there is a point of S^n fixed by the entire group and $\frac{\pi}{2}$ for all other reducible actions.

Orbifolds. If the group G is finite, then the resulting quotient space is a constant curvature one orbifold. In order to obtain a small diameter, a first guess would be to look at the resulting space corresponding to a large group action.

Example 3.1. Footballs in Dimension 2

Let $r = \begin{pmatrix} r_\theta & 0 \\ 0 & 1 \end{pmatrix}$ be a 3x3 real matrix, where r_θ is a 2x2 rotation matrix with rotation angle $\pi \cdot \frac{q}{p}$, a rational multiple of π with p and q are relatively prime. Then $S^2/ < r >= S^2/\mathbb{Z}_{2p}$ is a football of diameter π, with isotropy $\mathbb{Z}_{2p}$ at the cone points. By taking smaller rotations, we obtain thinner footballs. The quotient of S^2 by S^1, an infinite group, is a longitude of length π.

Example 3.2. Orbifold Lens Spaces in Dimension 3

Let $\Gamma = \mathbb{Z}_p$ be generated by $(z_1, z_2) \to (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$, where p and q are integers and z_1 and z_2 are complex numbers. Let $L(3,p,q) = S^3/\Gamma$. This is a (manifold) lens space when p and q are relatively prime. Otherwise, Γ does not act freely on S^3, and so $L(3,p,q)$ is an orbifold lens space. Since the z_1 complex plane is left invariant by Γ, we know that $\mathsf{diam}(\mathsf{L}(\mathsf{p},\mathsf{q}))$ is $\pi/2$ or π. The diameter is $\pi/2$ exactly when q is not an integer multiple of p, since there is no point of S^3 fixed by the entire group.

We now look at abelian groups, which provide useful intuition and are necessary in the proof of Theorem 3.14.

Example 3.3. S^n/Maximal Torus in $SO(n+1)$

Let T(n+1) be the maximal torus in SO(n+1). For n even, $\mathsf{diam}(\mathsf{S}^\mathsf{n}/\mathsf{T}(\mathsf{n}+1)) = \pi$, and for n odd, $\mathsf{diam}(\mathsf{S}^\mathsf{n}/\mathsf{T}(\mathsf{n}+1)) = \frac{\pi}{2}$

For n odd, the maximal torus T(n+1) consists of all n+1 by n+1 real matrices of the form $\begin{pmatrix} r_{\theta_1} & 0 & \dots & 0 \\ 0 & r_{\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r_{\theta_{\frac{n+1}{2}}} \end{pmatrix}$ (see Example 3.1). Then, $S^n/T(n+1)$ has diameter $\frac{\pi}{2}$ since there is no point of S^n fixed by the entire group, but there are invariant subspaces. For $n = 3$, $S^3/T(4)$ is an arc of length $\frac{\pi}{2}$. For larger, odd n, the quotient space $S^n/T(n+1)$ consists of a spherical polyhedron with $\frac{n+1}{2}$ vertices, where each edge has length $\frac{\pi}{2}$.

For n even, the maximal torus T(n+1) consists of all matrices of the form $\begin{pmatrix} r_{\theta_1} & 0 & \dots & 0 \\ 0 & r_{\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r_{\theta_{\frac{n}{2}}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then, $S^n/T(n+1)$ has diameter π since the last coordinate, x_{n+1}, is fixed by the entire group. For $n = 2$, as in Example 3.1, $S^2/T(3)$ is an arc of length π. For larger, even n, the quotient space $S^n/T(n+1)$ consists of a suspension of a polyhedron. Arcs of length π, which intersect at $(0, 0, ..., 1)$ and $(0, 0, ..., -1)$, are all $\frac{\pi}{2}$ away from each other.

The following is an example of a group which is abelian, but not contained in a maximal torus.

Example 3.4. The orbifold resulting from the group consisting of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice this group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is abelian and is not contained in a maximal torus S^1 of $SO(3)$. The x-axis is invariant under this action, but not fixed, so the action is reducible with resulting diameter $\frac{\pi}{2}$. A fundamental domain for the action is the orange peel wedge formed by half of the northern hemisphere, containing positive x and z. On the boundary, $(0, y, z)$ and $(0, -y, z)$ are identified, and $(x, y, 0)$ and $(x, -y, 0)$ are identified. The resulting orbifold is a 3-corner pillow with $\mathbb{Z}_2$ isotropy at all three points.

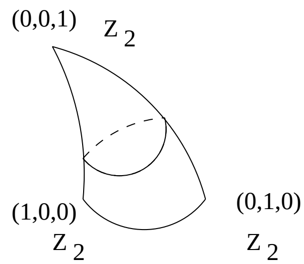

FIGURE 1. $S^2/\mathbb{Z}_2 \times \mathbb{Z}_2$

The following is an example of an irreducible action, which we know will have resulting diameter less than $\frac{\pi}{2}$ by Section 2.3.

Example 3.5. Three-Cornered Pillow Shaped Orbifold in Dimension 2

The icosahedral group I acts on S^2 giving rise to S^2/I, a three-cornered pillow with isotropy labeled below and diameter $\arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is approximately $\frac{\pi}{4.82}$ ([15],[20]).

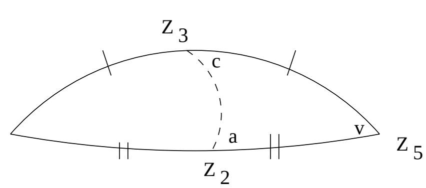

FIGURE 2. Icosahedral Fundamental Domain

Manifolds and the Diameter. If Γ acts properly discontinuously and freely, then S^n/Γ is a manifold. McGowan [25] has shown that diameters of these manifolds are bounded below by $\frac{1}{2}\arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is approximately $\frac{\pi}{9.63}$ (see also [18]). This lower bound is optimal and occurs in dimension $n = 3$, as follows.

First, note that a cyclic group $\mathbb{Z}_u$ acts on S^2 by fixing an axis of rotation, as in Example 3.1 The icosahedral group I also acts on S^2 as in Example 3.5. Pull back the group $\mathbb{Z}_u \times I$ in $SO(3) \times SO(3)$ into $SU(2) \times SU(2)$ to get $\mathbb{Z}_{2u} \times I^*$. This acts on S^3 by quaternionic left and right multiplication. $S^3/\mathbb{Z}_{2u} \times I^*$ is the desired manifold which has the above diameter bound. (Compare with Section 4.1.)

The diameters of all other manifolds S^n/Γ are computed exhaustively and found to be larger than this manifold. The diameters increase monotonically in n and approach $\frac{\pi}{2}$ as n approaches infinity.

3.2. **Irreducible Group Condition.** In this section we prove that a finite, irreducible group must satisfy a certain condition, which is independent of the dimension. In the process, we will prove some lemmas necessary for the next section. Notice that Lemmas 3.6 and 3.8 are true for any closed group, while finiteness is necessary in Lemma 3.9 and Theorem 3.11.

Lemma 3.6. *Let $G \subset O(n+1)$ be closed. If G is abelian then the action is reducible and so* $\mathsf{diam}(\mathsf{S^n}/\mathsf{G})$ *is $\frac{\pi}{2}$ or π.*

Proof of Lemma 3.6: After complexifying, G acts as a commuting set of unitary operators, and hence we can diagonalize all the matrices simultaneously over $\mathbb{C}$ via an orthonormal basis of eigenvectors. Let v_i be this orthonormal basis. Then, if $g \in G$, we know that $gv_i = e^{i\theta}v_i$. If $e^{i\theta}$ is real, then v_i generates a real invariant subspace. Otherwise, we also have $g\bar{v}_i = e^{-i\theta}\bar{v}_i$. Look at $\{v_i, \bar{v}_i\} = W \subset \mathbb{C}^{n+1}$ and $W \cap \mathbb{R}^{n+1}$. Now
$G(W \cap \mathbb{R}^{n+1}) \subset W \cap \mathbb{R}^{n+1}$. Also, $v_1 + \bar{v}_1, i(v_1 - \bar{v}_1) \in W$, and so W is a non-empty invariant subspace. Hence, G is reducible.□

Definition 3.7. In order to examine a neighborhood of the identity in $O(n+1)$, look at the **sup norm on** $GL(n+1)$, $\|M\| = sup\{|Mx| \mid x \in \mathbb{R}^{n+1} \text{ and } |x| = 1\}$, where $|x|$ is the Euclidean norm. Then define an epsilon ball about M in $O(n+1)$ as $B_\epsilon(\mathrm{Id}) = \{M \in O(n+1) \mid \|\mathrm{Id} - M\| < \epsilon\}$

Lemma 3.8. *Consider $U = B_{1/2}(Id)$. This satisfies* [9]

1)*Let $g \in O(n+1), h \in U, [g,[g,h]] = id \Rightarrow [g,h] = id$.*
2)*$g, h \in U \Rightarrow$ the sequence $g_0 = g,\ g_1 = [g,h],\ g_2 = [g,[g,h]] = [g, g_1], \dots, g_n, \dots$ converges to $id \in O(n+1)$.*

In [9], this lemma is only proven for $U = B_{1/4}(\mathrm{Id})$, but one can easily modify the proof to work for $U = B_{1/2-\epsilon}(\mathrm{Id})$, for any $\epsilon > 0$. To show that this holds for $U = B_{1/2}(\mathrm{Id})$, it is necessary to show that 2) holds. Assume for contradiction that 2) does not hold for $U = B_{1/2}(\mathrm{Id})$. Choose $g, h \in U$ which violate 2). Now, g, h must be in $B_{1/2-\epsilon}(\mathrm{Id})$ for some $\epsilon > 0$, and so 2) does not hold for $B_{1/2-\epsilon}(\mathrm{Id})$, a contradiction.

Lemma 3.9. *Let Γ be a finite group in $O(n+1)$. If $\Gamma \subset U = B_{1/2}(Id)$, then Γ is abelian.*

Proof of Lemma 3.9: Let $g, h \in \Gamma$. We know that Γ is discrete since Γ is finite. Since $g_i \to \mathrm{id}$ in Lemma 3.8, then $g_i = \mathrm{id}$ for some i. Applying 1 from the lemma repeatedly, we see that $g_1 = [g,h] = \mathrm{id}$. Thus, Γ is abelian. □

Definition 3.10. Let $p \in S^n$ be a point with a trivial isotropy group for a discrete action of Γ on S^n (such points exist as in [22, page 28]). A **Dirichlet fundamental domain** centered at p is the set $\{x \in S^n \mid d(x,p) \leq d(x,gp), \forall g \in \Gamma\}$. Since Γ is discrete, it is finite, so label the elements as $\gamma_1, ..., \gamma_m$. Let
$H_{\gamma_i} = \{x \in S^n \mid d(x,p) \leq d(x, \gamma_i(p)\}$. Then, the Dirichlet fundamental domain is $\cap H_{\gamma_i}$ which is a fundamental domain for the action of Γ on S^n [22, pages 29–30].

Theorem 3.11. *Given a Dirichlet fundamental domain, each co-dimension one face on the boundary arises from the half-space between the center point and its image under a group element. If the action is irreducible and finite, then one of these domain generating group elements moves some point y in S^n at least* $\arccos(\frac{7}{8})$*, which is approximately $\frac{\pi}{6.2}$, away from y.*

Proof of Theorem 3.11: Let Γ be finite, and let the action of Γ be irreducible. Then, looking at the sup norm on $O(n+1)$, we will show that $\|\mathrm{Id} - g_i\| \geq \frac{1}{2}$ for some i, where g_i generates the Dirichlet fundamental domain.

Assume not for contradiction. Then all the g_i are in $B_{1/2}(\mathrm{Id})$, the ball of radius $1/2$ about the identity. Applying Lemma 3.9, we see that all the g_i commute. Since these generate the group, then the entire group is abelian. By Lemma 3.6, the action is reducible, a contradiction. Thus, by definition of the sup norm, the Euclidean distance between y and $g_i(y)$ must be greater than or equal to $1/2$ for some i and for some $y \in S^n$. For any such i and y, form the triangle consisting of the vectors y and $g_i(y)$ through the point $(0, ..., 0)$ and angle θ between them, and with c as the side opposite the angle θ. Using trigonometry, we see that $1/4 \leq c^2 = 2 - 2\cos\theta$ or $2\cos\theta \leq 7/4$. Hence, $\theta \leq \arccos 7/8$. Converting back to spherical distance, which is the Euclidean angle between vectors, we see that $d(y, i(y)) \geq \arccos 7/8$, as desired.□

Remark 3.12. This condition on the group is independent of the dimension.

3.3. **Explicit Lower Bound Given a Fixed Dimension.** In this section, we will prove that given a fixed dimension, there exists a lower bound on the diameter resulting from finite groups. This lower bound is explicit and depends only on the dimension. While finiteness is used in the proof of Theorem 3.14, notice that Lemma 3.13 holds for any closed, non-transitive group $G \subset O(n+1)$.

Lemma 3.13. *Let $G \subset O(n+1)$ be closed and non-transitive. If G' has index k in G then* $\mathsf{diam}(\mathsf{S^n/G}) \geq \frac{\mathsf{diam}(\mathsf{S^n/G'})}{\mathsf{2(k-1)}}$ *(compare with* [18, page 103]*).*

Proof of Lemma 3.13 Let d be the diameter of S^n/G and let $p \in S^n/G$. Now $B_d(p)$, the ball of radius d about p, must cover S^n/G. Since the index of G' in G is k, notice that S^n/G is a k-fold cover of S^n/G' and so p must lift to $p'_1, ..., p'_k$ via the coset map. Hence $B_d(p)$ must lift to $B_d(p'_1), B_d(p'_2), ..., B_d(p'_k)$, where $p'_i \in S^n/G'$. These balls cover S^n/G'. Let p' be in the intersection of any two balls in S^n/G' and let $q' \in S^n/G'$. To show that $d(p', q') \leq 2(k-1)d$, create a path from p' to q' by traveling through the $p'_i s$, the centers of the balls.

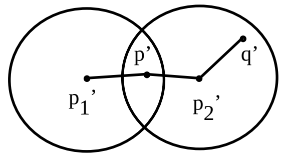

FIGURE 3. Index 2

The length of this path is at most $2(k-1)d$, since if you hit any ball more than once, you will obtain a loop, which can be discarded. Notice that p was arbitrary, so that given any $p' \in S^n/G'$, we can find some point in S^n/G and repeat the process of lifting so that p' arises in the intersection of two balls as above. Hence $d(p', q') \leq 2(k-1)d$ for any points $p', q' \in S^n/G'$, and so $\mathsf{diam}(\mathsf{S^n/G'}) \leq 2(\mathsf{k}-1)\mathsf{d}$, as desired.□

Corollary 3.13.1. *If $G' \in SO(n+1)$ with $\mathsf{diam}(\mathsf{S^n/G'}) = \mathsf{d'}$, and $G \in O(n+1)$ is a 2-fold extension of G' then $\mathsf{diam}(\mathsf{S^n/G}) \geq \frac{\mathsf{d'}}{2}$.*

Theorem 3.14. *Let $\Gamma \subset O(n+1)$ be finite. There exists ϵ depending only on n so that $\mathsf{diam}(\mathsf{S^n}/\Gamma) \geq \epsilon(\mathsf{n})$.*

Proof of Theorem 3.14: Let $\Gamma \subset O(n+1)$ be finite. Let Γ' be the subgroup generated by $\Gamma \cap U \subset B_{1/2}(\mathrm{Id})$, which is abelian by Lemma 3.6.

We will universally bound the index of Γ' in Γ with a constant depending only on the dimension n. Write $\Gamma = \delta_1\Gamma' \cup \delta_2\Gamma' \cup ... \cup \delta_k\Gamma'$ where $\delta_i\Gamma'$ are distinct cosets, as Γ is finite. We will find a bound on k. Note that if $\|\delta_i - \delta_j\| < \frac{1}{2}$, then
$\|\delta_j^{-1}\delta_i - \mathrm{Id}\| = \|\delta_j^{-1}(\delta_i - \delta_j)\| = \|\delta_i - \delta_j\| < \frac{1}{2}$, since $\delta_j^{-1} \in O(n+1)$. Hence, $\delta_j^{-1}\delta_i \in U \cap \Gamma'$. Therefore, $\delta_i = \delta_j(\delta_j^{-1}\delta_i) \in \delta_j\Gamma'$, and so δ_i and δ_j are in the same coset. Since we have written Γ as a union of distinct cosets, then the δ_is are all at least $\frac{1}{2}$ away from each other in $O(n+1)$. Place disjoint $\frac{1}{2}$ balls in $SO(n+1)$ about each δ_i. Now k, the index of Γ' in Γ must be less than or equal to the maximum number of disjoint $\frac{1}{2}$ balls we can put in $SO(n+1)$. Using volume estimates, it follows that k is bounded above by a constant depending only on the dimension n, call it k.

Finally, we will find a lower bound for the diameter of S^n/Γ. Since Γ' is abelian, we know that the diameter of S^n/Γ' is $\frac{\pi}{2}$ or π, by Lemma 3.6. Let d be the diameter of S^n/Γ. From Lemma 3.13 we know that $d \geq \frac{\pi}{4(k-1)}$, where k depends only on n, as desired. □

Corollary 3.14.1. *If Γ is a finite subgroup of the isometry group of M^n, where M^n is $\mathbb{C}P^n$ or $\mathbb{H}P^n$, then there exists ϵ depending only on n, so that $\mathsf{diam}(\mathsf{M^n}/\Gamma) \geq \epsilon(\mathsf{n})$.*

Proof of Corollary 3.14.1: Fix n. The proof is an extension of Theorem 3.14 via a Hopf-fibration argument.

For $\mathbb{C}P^n$, let $\Gamma_i^c \subset \mathrm{Isom}(\mathbb{C}P^n) = SU(n+1)/\mathbb{Z}_{n+1} \cup cSU(n+1)/\mathbb{Z}_{n+1}$, where c is complex conjugation, be finite. Recall that $\mathbb{C}P^n$ is isometric to $S^{2n+1}/U(1)$ via the submersion metric. One obtains the Hopf-fibration from the diagonal embedding of

$U(1)$ into $SU(n+1)$. Now, look at the inverse image of Γ_i^c in $O(2n+2)$ via lifting to $SU(n+1) \cup cSU(n+1)$, and call it Γ_i. Find a finite abelian subgroup of Γ_i as in in Theorem 3.14, and call it Γ_i'. We know that the index, k, of Γ_i' in Γ_i depends only on n. Diagonalize the Γ_i matrices over $\mathbb{C}$. We will now apply $U(1)$ from the Hopf fibration. Notice that $\Gamma_i' \cdot U(1)$ will also have index k in $\Gamma_i \cdot U(1)$. Since Γ_i and $U(1)$ are all diagonal, they preserve an axis. Hence, they are reducible. Then, $\mathsf{diam}(\mathsf{S}^{2n+1}/\Gamma_i' \cdot \mathsf{U}(1))$ is at least $\frac{\pi}{2}$. We use the index k, depending only on n, to obtain a lower bound on the diameter of $S^{2n+1}/\Gamma_i \cdot U(1) = (S^{2n+1}/U(1))/\Gamma_i = \mathbb{C}P^n/(\Gamma_i/\mathbb{Z}_{n+1})$, as desired.

For $\mathbb{H}P^n$, let $\Gamma_i^h \subset \mathrm{Isom}(\mathbb{H}P^n) = Sp(n+1)/\mathbb{Z}_2$, be finite. The Hopf fibration $S^{4n+3} \to \mathbb{H}P^n$ embeds $Sp(1)$ into $Sp(n+1) \subset SO(4n+4)$. The argument follows by pulling Γ_i^h back into $SO(4n+4)$ and continuing as above. Once diagonalizing $\Gamma_i^{'}$, we see that $\Gamma_i^{'}$ and $Sp(1)$ preserve an axis since $\Gamma_i^{'}$ acts on the left and $Sp(1)$ acts on the right. Hence, the action is reducible and the argument follows as above. □

3.4. **Coxeter Groups.** In this section, we will prove that if Γ is a Coxeter group, a group generated by reflections, (see [24] and [22] for background information) then $\mathsf{diam}(\mathsf{S}^n/\Gamma) \geq \frac{\pi}{8.10}$.

The Weyl group of a Lie group is a Coxeter group, so Coxeter groups are of natural interest. To examine the meaning of the diameter lower bound for Lie groups, let G be a compact Lie group of dimension $n+1$. Look at a compact torus T in G. On the Lie algebra level, $\mathfrak{t} \subset \mathfrak{g}$, as a maximal abelian subalgebra. Let $N(T)$ be the normalizer of T. Now, the Weyl group, $W = N(T)/T$ acts on $\mathfrak{t}$ via conjugation. Let $Ad(G)$ be the adjoint action of G on $\mathfrak{g}$. There exists a bi-invariant metric on the Lie algebra so that $Ad(G)$ acts by isometries. Examine $S^n(1) \subset \mathfrak{g}$, where $S^n(1)$ has radius one. Now, $S^n(1)/Ad(G) = (S^n(1) \cap \mathfrak{t})/W$. Hence, a lower bound on the diameter of $(S^n(1) \cap \mathfrak{t})/W$ gives a lower bound on the diameter of $S^n(1)/Ad(G)$.

Notation and Background. Let r be a non-trivial vector in $\mathbb{R}^{n+1}$ and define H_r as the subspace orthogonal to r. A reflection R_r in $\mathbb{R}^{n+1}$ sends r to $-r$, fixes H_r and sends any vector v to

$$v - \frac{< v, r >}{< r, r >} r.$$

Notice that $R_r \in O(n+1)$. A Coxeter group Γ is a finite group generated by reflections.

A root system R for Γ is a finite set of nonzero vectors in $\mathbb{R}^{n+1}$ so that each vector r, called a root, satisfies:

$$1) \forall c \in \mathbb{R}, \mathrm{R} \cap cr = \{r, -r\}$$

$$2) R_r(\mathrm{R}) = \mathrm{R}.$$

A simple system Δ, with simple roots r_i, is a real basis for the root system so that each root vector in R is a linear combination of simple roots where the coefficients all have the same sign. Every Coxeter group has a simple system [24], so let $\Delta_\Gamma = \{r_1, r_2, ..., r_m\}$, where r_i is a simple root of the Coxeter group Γ. Γ is generated by the reflections R_{r_i} [22]. Let

$$D = \{v \in \mathbb{R}^{n+1} \text{ so that } < v, r_i > \geq 0 \text{ for all simple } r_i\}.$$

D is a fundamental domain for Γ and there are no further identifications on the boundary of D [24, pages 22–23]. A dual basis q_i for r_i, where $< q_i, r_j >= \delta_{ij}$, forms the vertices of the fundamental domain [22]. Hence, $D \cap S^n$ is a fundamental domain for the action of Γ on S^n.

All irreducible Coxeter groups, groups for which Δ_Γ is not the union of two nonempty orthogonal subsets, are classified in [22] via the Coxeter diagram and its corresponding simple roots ([22, page 71]). These are $H_n^2, G_2, I_3, I_4, F_4, E_6, E_7, E_8, A_n, B_n$, and D_n. We use the simple roots r_i to compute a dual basis q_i as follows. Let A be the matrix with $< r_i, r_j >$ as the ijth entry, and let $A^{-1}_{(ij)}$ be the ijth entry of A^{-1}. Then

$$q_i = \sum_{j=1}^{3} A^{-1}_{(ij)} r_j$$

is a dual basis vector for Γ [22, page 52]. Since the dual basis vectors q_i are the vertices of the fundamental domain, and the fundamental domain has no additional identifications on it, the diameter of S^n/Γ is achieved as the largest spherical distance between dual basis vectors. Since spherical distance is the angle, we look for the two basis vectors with maximum angle between them. After a description of each group and the dual basis vectors, we give the two vectors which achieve the diameter, and also give the value for the diameter. Notice that we eliminate the dihedral group H_n^2 and G_2 from the list because, as Coxeter groups, they act on $\mathbb{R}^2$.

Define $\alpha = 2\cos\frac{\pi}{5}$ and $\beta = \cos\frac{2\pi}{5}$.

I_3. I_3 has order $120 = 2^3 3 \cdot 5$. It acts on S^2 as I^-, the Coxeter extension of I, the icosahedral group. The simple root vectors are

$$\begin{aligned} &r_1 = [\beta\alpha + \beta, \beta, -\beta\alpha], \quad r_2 = [-\beta\alpha - \beta, \beta, \beta\alpha], \\ &r_3 = [\beta\alpha, -\beta\alpha - \beta, \beta]. \end{aligned}$$

The dual basis vectors are

$$\begin{aligned} &q_1 = [-\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}], \quad q_2 = [1, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{2}{-3+\sqrt{5}}], \\ &q_3 = [1, 0, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}]. \end{aligned}$$

The diameter of $\frac{S^2}{I_3}$ is achieved by q_1 and q_3 and is $\arccos\frac{1}{3}\frac{\sqrt{3}}{\sqrt{5-2\sqrt{5}}} = \arccos\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}} \approx \frac{\pi}{4.82}$.

I_4. I_4 has order $120^2 = 2^6 3^2 5^2$. It acts on S^3 as $(I^* \times I^*)^-$, the Coxeter extension of $I^* \times I^*$ (see [14, page 57]), where I^* is as in Example 3.1. The simple root vectors are

$$r_1 = [\beta\alpha + \beta, \beta, -\beta\alpha, 0], \quad r_2 = [-\beta\alpha - \beta, \beta, \beta\alpha, 0],$$
$$r_3 = [\beta\alpha, -\beta\alpha - \beta, \beta, 0], \quad r_4 = [-\beta\alpha, 0, -\beta\alpha - \beta, \beta].$$

The dual basis vectors are

$$q_1 = [-\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{2}{-7+3\sqrt{5}}], \quad q_2 = [1, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{2}{-3+\sqrt{5}}, -\tfrac{2\sqrt{5}-2}{-7+3\sqrt{5}}],$$
$$q_3 = [1, 0, -\tfrac{\sqrt{5}-1}{-3+\sqrt{5}}, -\tfrac{3\sqrt{5}-5}{-7+3\sqrt{5}}], \quad q_4 = [0, 0, 0, \tfrac{-4\sqrt{5}-2}{-7+3\sqrt{5}}].$$

The diameter of $\frac{S^3}{I_4}$ is achieved by q_1 and q_4 and is $\pi - \arccos \sqrt{2\frac{-9+4\sqrt{5}}{(-7+3\sqrt{5})^2}} \approx \frac{\pi}{8.10}$.

F_4. F_4 has order $2^7 3^2$. It acts on S^3 as follows. This is the group of symmetries of a regular solid in $\mathbb{R}^4$ having 24 (three-dimensional) faces which are octahedra [11]. The simple root vectors are

$$r_1 = [\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}], \quad r_2 = [1, 0, 0, 0],$$
$$r_3 = [-1, 1, 0, 0], \quad r_4 = [0, -1, 1, 0].$$

The dual basis vectors are

$$q_1 = [0, 0, 0, -2], \quad q_2 = [1, 1, 1, -3],$$
$$q_3 = [0, 1, 1, -2], \quad q_4 = [0, 0, 1, -1].$$

The diameter of $\frac{S^3}{F_4}$ is achieved by q_1 and q_4 and is $\frac{\pi}{4}$.

E_6. E_6 has order $2^7 3^4 5$ and acts on S^5. It is the group of automorphisms of a configuration of 27 lines on a cubic surface. The simple root vectors are

$$r_1 = [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}], \quad r_2 = [-1, 1, 0, 0, 0, 0, 0, 0],$$
$$r_3 = [0, -1, 1, 0, 0, 0, 0, 0], \quad r_4 = [0, 0, -1, 1, 0, 0, 0, 0],$$
$$r_5 = [0, 0, 0, -1, 1, 0, 0, 0], \quad r_6 = [0, 0, 0, 0, -1, 1, 0, 0].$$

The dual basis vectors are

$$q_1 = [0, 0, 0, 0, 0, 0, -1, -1], \quad q_2 = [\tfrac{-5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{-1}{2}, \tfrac{-1}{2}],$$
$$q_3 = [\tfrac{-2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -1, -1], \quad q_4 = [\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-3}{2}, \tfrac{-3}{2}],$$
$$q_5 = [\tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{2}{3}, -1, -1], \quad q_6 = [\tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{5}{6}, \tfrac{-1}{2}, \tfrac{-1}{2}].$$

The diameter of $\frac{S^5}{E_6}$ is achieved by q_2 and q_6 and is $\frac{\pi}{3}$.

E_7. E_7 has order $2^{10} 3^4 5 \cdot 7$ and acts on S^6. The simple root vectors are

$$r_1 = [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}], \quad r_2 = [-1, 1, 0, 0, 0, 0, 0, 0],$$
$$r_3 = [0, -1, 1, 0, 0, 0, 0, 0], \quad r_4 = [0, 0, -1, 1, 0, 0, 0, 0],$$
$$r_5 = [0, 0, 0, -1, 1, 0, 0, 0], \quad r_6 = [0, 0, 0, 0, -1, 1, 0, 0],$$
$$r_7 = [0, 0, 0, 0, 0, -1, 1, 0].$$

The dual basis vectors are

$$\begin{array}{ll} q_1 = [\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-7}{4}], & q_2 = [-1, 0, 0, 0, 0, 0, 0, -1], \\ q_3 = [-1, -1, 0, 0, 0, 0, 0, -2], & q_4 = [-1, -1, 0, 0, 0, 0, 0, -3], \\ q_5 = [\frac{-3}{4}, \frac{-3}{4}, \frac{-3}{4}, \frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{-9}{4}], & q_6 = [\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-3}{2}], \\ q_7 = [\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{3}{4}, \frac{-3}{4}]. & \end{array}$$

The diameter of $\frac{S^6}{E_7}$ is achieved by q_2 and q_7 and is $\arccos \frac{\sqrt{3}}{3} \approx \frac{\pi}{3.29}$.

$\boldsymbol{E_8}$. E_8 has dimension $2^{14}3^5 5^2 7$ and acts on S^7. The simple root vectors are

$$\begin{array}{ll} r_1 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}], & r_2 = [-1, 1, 0, 0, 0, 0, 0, 0], \\ r_3 = [0, -1, 1, 0, 0, 0, 0, 0], & r_4 = [0, 0, -1, 1, 0, 0, 0, 0], \\ r_5 = [0, 0, 0, -1, 1, 0, 0, 0], & r_6 = [0, 0, 0, 0, -1, 1, 0, 0], \\ r_7 = [0, 0, 0, 0, 0, -1, 1, 0], & r_8 = [0, 0, 0, 0, 0, 0, -1, 1]. \end{array}$$

The dual basis vectors are

$$\begin{array}{ll} q_1 = [-1, -1, -1, -1, -1, -1, -1, -1], & q_2 = [\frac{-3}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}], \\ q_3 = [-2, -2, -1, -1, -1, -1, -1, -1], & q_4 = [\frac{-5}{2}, \frac{-5}{2}, \frac{-5}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}], \\ q_5 = [-2, -2, -2, -2, -1, -1, -1, -1], & q_6 = [\frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}], \\ q_7 = [-1, -1, -1, -1, -1, -1, 0, 0], & q_8 = [\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}]. \end{array}$$

The diameter of $\frac{S^7}{E_8}$ is achieved by q_2 and q_8 and is $\frac{\pi}{4}$.

$\boldsymbol{A_n}$. A_n has order $(n+1)!$ and acts on S^{n-1} as follows. Consider the symmetric group S_{n+1} acting as permutations on the coordinates of $\mathbb{R}^{n+1}$. Notice that S_{n+1} fixes the line corresponding to $e_1 + e_2 + ... + e_{n+1}$, where e_i is the standard basis vector in $\mathbb{R}^{n+1}$. The orthogonal plane consisting of vectors whose coordinates add up to 0 is left invariant under S_{n+1}. Let A_n be the action of S_{n+1} restricted to this orthogonal plane. A_n fixes the origin in this new $\mathbb{R}^n$, and so it acts on S^{n-1}. Notice that A_n is not the group of even permutations. The simple root vectors are

$$\begin{aligned} r_1 &= [-1, 1, 0, \dots, 0, 0, 0], \\ r_2 &= [0, -1, 1, 0, \dots, 0, 0], \\ r_3 &= [0, 0, -1, 1, 0, \dots, 0], \\ &\vdots \\ r_n &= [0, 0, 0, 0, \dots, -1, 1]. \end{aligned}$$

The dual basis vectors are

$$\begin{aligned} q_1 &= [\tfrac{-n}{n+1}, \tfrac{1}{n+1}, \tfrac{1}{n+1}, \dots, \tfrac{1}{n+1}], \\ q_2 &= [\tfrac{-(n-1)}{n+1}, \tfrac{-(n-1)}{n+1}, \tfrac{2}{n+1}, \dots, \tfrac{2}{n+1}], \\ q_3 &= [\tfrac{-(n-2)}{n+1}, \tfrac{(n-2)}{n+1}, \tfrac{(n-2)}{n+1}, \tfrac{3}{n+1}, \dots, \tfrac{3}{n+1}], \\ &\vdots \\ q_{n-1} &= [\tfrac{-2}{n+1}, \tfrac{-2}{n+1}, \dots, \tfrac{-2}{n+1}, \tfrac{n-1}{n+1}, \tfrac{n-1}{n+1}], \\ q_n &= [\tfrac{-1}{n+1}, \tfrac{-1}{n+1}, \dots, \tfrac{-1}{n+1}, \tfrac{n}{n+1}]. \end{aligned}$$

The diameter of $\frac{S^{n-1}}{A_n}$ is achieved by q_1 and q_n and is $\arccos\frac{1}{n}$, which goes to $\frac{\pi}{2}$ as $n \to \infty$. The diameter of $\frac{S^2}{A_3}$ is $\arccos\frac{1}{3} \approx \frac{\pi}{2.56}$. A_3 is the group T^-, the full isometry group of a regular tetrahedron including reflection symmetries, as in [20, pages 18–20].

$\boldsymbol{B_n}$. B_n has order $2^n n!$ and acts on S^{n-1} as follows. Let S_n be as in the description of A_n above. The sign change reflections sending e_i to its negative and fixing all other e_j generate a group of order 2^n isomorphic to $\mathbb{Z}_2^n$. Conjugating a sign change by a transposition will yield another sign change. B_n is the semi-direct product of S_n and $\mathbb{Z}_2^n$. The simple root vectors are

$$\begin{aligned}
r_1 &= [1, 0, 0, \ldots, 0, 0, 0],\\
r_2 &= [-1, 1, 0, \ldots, 0, 0, 0],\\
r_3 &= [0, -1, 1, 0, \ldots, 0, 0],\\
r_4 &= [0, 0, -1, 1, 0, \ldots, 0],\\
&\vdots\\
r_n &= [0, 0, 0, 0, \ldots, -1, 1].
\end{aligned}$$

The dual basis vectors are

$$\begin{aligned}
q_1 &= [1, 1, 1, \ldots, 1, 1, 1],\\
q_2 &= [0, 1, 1, \ldots, 1, 1, 1],\\
q_3 &= [0, 0, 1, \ldots, 1, 1, 1],\\
&\vdots\\
q_n &= [0, 0, 0, \ldots, 0, 0, 1].
\end{aligned}$$

The diameter of $\frac{S^{n-1}}{B_n}$ is achieved by q_1 and q_n and is $\arccos\frac{\sqrt{n}}{n}$. As $n \to \infty$, the diameter approaches $\frac{\pi}{2}$. The diameter of $\frac{S^2}{B_3}$, is $\arccos\frac{\sqrt{3}}{3} \approx \frac{\pi}{3.29}$. B_3 is the group O^-, the orthogonal extension of the octahedral group [20, pages 20–21].

$\boldsymbol{D_n}$. D_n has order $2^{n-1}n!$ and acts on S^{n-1}. D_n is a subgroup of index 2 in B_n. Look at the subgroup of $\mathbb{Z}_2^n$ consisting only of sign changes which involve an even number of signs, which is generated by $e_i + e_j \to -(e_i + e_j)$ for $i \neq j$. D_n is the semi-direct product of this subgroup with S_n. The simple root vectors are

$$\begin{aligned}
r_1 &= [1, 1, 0, \ldots, 0, 0, 0],\\
r_2 &= [-1, 1, 0, \ldots, 0, 0, 0],\\
r_3 &= [0, -1, 1, 0, \ldots, 0, 0],\\
&\vdots\\
r_n &= [0, 0, 0, 0, \ldots, -1, 1].
\end{aligned}$$

The dual basis vectors are

$$\begin{aligned} q_1 &= [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}], \\ q_2 &= [\tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}], \\ q_3 &= [0, 0, 1, 1, \dots, 1, 1], \\ q_4 &= [0, 0, 0, 1, \dots, 1, 1], \\ &\vdots \\ q_n &= [0, 0, 0, 0, \dots, 0, 1]. \end{aligned}$$

For $n \geq 4$, the diameter of $\frac{S^{n-1}}{D_n}$ is achieved by q_1 and q_n and is $\arccos \frac{\sqrt{n}}{n}$. As $n \to \infty$, the diameter approaches $\frac{\pi}{2}$. For $n = 3$, the diameter of $\frac{S^2}{D_3}$ is achieved by q_1 and q_2 and is $\arccos \frac{1}{3} \approx \frac{\pi}{2.56}$. Notice that $D_3 (= A_3)$ is the group T^- as described above.

Summary of Coxeter Orbifold Diameter Results		
Coxeter Group	*Other Group Descriptions (compare with* [14, pages 57–61], [16, 12], *and* [22, page 81]*)*	*Resulting Diameter Lower Bound*
I_3	Icosahedral I^-	$\pi/4.82$
I_4	$(I^* \times I^*)^-$	$\pi/8.10$
F_4	Index-2 extension of $O^* \times_{C_2} O^*$	$\pi/4$
E_6	Automorphisms of 27 lines on a cubic $\mathbb{P}^2_6$	$\pi/3$
E_7	Automorphisms of 56 lines on $\mathbb{P}^2_7$	$\pi/3.29$
E_8	Automorphisms of 240 lines on $\mathbb{P}^2_8$	$\pi/4$
A_n		$\arccos 1/n$
B_n		$\arccos \sqrt{n}/n$
$D_n, n > 3$		$\arccos \sqrt{n}/n$
$A_3 = D_3$	Tetrahedral T^-	$\pi/2.56$
B_3	Octahedral O^-	$\pi/3.29$
A_4	Index-2 extension of $I^* \times_{\check{I}} I^{*1}$	$\pi/2.38$
B_4	$(O^* \times_{D_3} O^*)^-$	$\pi/3$
D_4	Index-2 extension of $T^* \times_{C_3} T^*$	$\pi/3$

TABLE 1. Summary of Coxeter Orbifold Diameter Results

Lower Bound for Coxeter Groups.

Theorem 3.15. *If Γ is a Coxeter group, generated by reflections, then* $\mathsf{diam}(\mathsf{S}^n/\Gamma) \geq \frac{\pi}{8.10}$.

Proof of Theorem 3.15: If Γ is a reducible Coxeter group, then Δ_Γ is the union of two non-empty orthogonal subsets A and B by definition. Look at the subspaces H_A and H_B generated by A and B, respectively. We claim these subspaces are invariant under Γ. Notice that for any $b \in B, R_b$ fixes H_A by definition of R_b, since $H_A \subset H_b$. Now $H_A^\perp = H_B$ and Γ acts by isometries. Hence, H_b leaves H_B invariant also. Similarly, H_a leaves both H_A and H_B invariant. Since, Γ is reducible in the sense of invariant subspaces, we know that the diameter of $S^n/\Gamma \geq \frac{\pi}{2}$.

For every irreducible Coxeter group, we have found a dual basis and then exhaustively computed the diameter of S^n/Γ. Notice that the lowest diameter occurs in dimension three. This is achieved as the diameter of the quotient of S^3 by I_4, the Coxeter group extension of $I^* \times I^*$. Given a fixed dimension n which is large, A_{n+1}, B_{n+1} and D_{n+1} are the only Coxeter groups which act on S^n. Since A_{n+1} has a resulting diameter of $\arccos \frac{1}{n+1}$, while B_{n+1} and D_{n+1} each have resulting diameter $\arccos \frac{\sqrt{n+1}}{n+1}$, we see that the smallest diameter is achieved by both B_{n+1} and D_{n+1}. This smallest diameter, $\arccos \frac{\sqrt{n+1}}{n+1}$, increases monotonically in n. As n approaches infinity, the diameter approaches $\frac{\pi}{2}$. $\square$

4. Infinite Groups

4.1. **Intuition and Examples.**

Cohomogeneity-One Actions and their Resulting Diameter. When the quotient space is an interval, the action is called a cohomogeneity-one action. In Example 3.1, we saw that the quotient of S^2 by S^1, the maximal torus of SO(3), is a longitude of length π. In general, since the orbits Gp are isoparametric hypersurfaces in spheres, it is well known that the length of the intervals are $\frac{\pi}{p}$ where $p = 2, 3, 4$ or 6 ([23]). Hence, the diameter is at least $\frac{\pi}{6}$. There are only two examples, in dimensions 7 and 13, where the diameter is equal to $\frac{\pi}{6}$. However, unlike the manifold and Coxeter orbifold cases, there is an entire class of cohomogeneity-one actions on S^n, including actions for arbitrarily large dimensions, which result in a quotient space of diameter $\frac{\pi}{4}$.

Arbitrary Actions. If the group is infinite, then the resulting quotient space is an Alexandrov space with curvature bounded below. The explicit orbifold lower bounds in Theorem 1 do not apply since discreteness was needed in the proof.

Example 4.1. $\mathbb{C}P^n$

Look at the Hopf action on S^{2n+1}. Then $S^{2n+1}/S^1 = \mathbb{C}P^n$ has diameter $\frac{\pi}{2}$.

Example 4.2. $S^3/S^1 \times I^*$

As in Section 3.1, $S^1 \times I^*$ acts on S^3 by quaternionic left and right multiplication. Now,

$$\mathsf{diam}(\mathsf{S}^3/\mathsf{S}^1 \times \mathsf{I}^*) = \mathsf{diam}(\mathbb{C}\mathsf{P}^1/\mathsf{I}) = \mathsf{diam}(\mathsf{S}^2(1/2)/\mathsf{I}),$$

where $S^2(1/2)$ is the 2-sphere of radius 1/2. Since
$\mathsf{diam}(\mathsf{S}^2/\mathsf{I}) = \arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is approximately $\frac{\pi}{4.82}$, we see that
$\mathsf{diam}(\mathsf{S}^2(1/2)/\mathsf{I}) = .5\arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is approximately $\frac{\pi}{9.63}$.

4.2. **Infinite Group Conditions.** Let G be infinite. If G is transitive, then S^n/G is a point, which has diameter 0. Hence, we restrict to non-transitive actions.

The diameter resulting from a group is smaller than or equal to the diameter resulting from its subgroups, so for any subgroup, we restrict to its closure. This restriction is important, since if G is not closed, then the quotient is not even Hausdorff. For example, look at the group $G \subset SO(3)$ acting on S^2 generated as follows. Let r_θ be a 2x2 real rotation matrix with rotation angle an irrational multiple of π. Let G be generated by $\begin{pmatrix} r_\theta & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & r_\theta \end{pmatrix}$. This action is not transitive on S^2 but any point gets arbitrarily close to any other point.

In general, given generators, it can be quite difficult to decide if the resulting group is finite or infinite. For example, for the two generators above, let θ be a rational multiple of π. Notice that G is not cyclic or dihedral. The largest order of any irreducible, finite group in $O(3)$ acting on S^2 is $60 = |I|$ [1]. Hence, if θ is small enough, then G is infinite.

4.3. **Existence of a Lower Bound in a Fixed Dimension.** In this section we will prove the following theorem:

Theorem 4.3. *If G is a non-transitive group, then there exists ϵ, depending only on n, so that* $\mathsf{diam}(\mathsf{S}^\mathsf{n}/\mathsf{G}) \geq \epsilon(\mathsf{n})$.

The lower bounds are not explicit since the proof is by contradiction.

In the proof, we show that a non-transitive sequence of groups cannot converge to a transitive subgroup of $SO(n+1)$. The following example is good to keep in mind while reading the details of the proof, since we must show that this behavior cannot occur on the sphere: (Compare with Lemma 4.9.)

Example 4.4. Let T^2 be a torus. Let G_i be lines of rational slope converging (in the sense of Lemma 4.5) to a line of irrational slope.

Now, G_i are non-transitive groups converging to a transitive group on T^2. Note that the corresponding Lie algebras are all dimension 1, and as subspaces, they converge to a dimension 1 space. The Lie algebras are all abelian.

Lemma 4.5. *Let G be a compact Lie group, and let G_i be a sequence of closed subgroups of G. Then*

1. *There exists a subsequence and a compact group G_∞ such that*
   (a) $G_\infty = \{$*limit of any convergent sequence* g_i, *where* $g_i \in G_i$ *for all* $i.\}$
   (b) G_∞ *does not get any larger when a smaller subsequence is taken.*
   (c) *for every neighborhood U of G_∞, there exists i_o so that $G_i \subset U$ for $i \geq i_o$.*

2. *There exists a further subsequence so that G_i is conjugate to $G_i' \subset G_\infty$ in G, where G_i' converges to G_∞ in the sense of a).*

Notation:
We say that $G_i \to G_\infty$ if G_i converges to G_∞ in the sense of 1.
We say that $G_i \hookrightarrow G_\infty$ if $G_i \to G_\infty$ and $G_i \subset G_\infty$, as in 2.
In general, for a Lie group H, let H^o be the connected component of the identity of H. Denote $Z(H)$ as the center of H, and $\mathfrak{Z}(\mathfrak{h})$ as the center of the corresponding Lie algebra $\mathfrak{h}$. Let $\mathfrak{h}^{ss}$ be the semi-simple part of $\mathfrak{h}$. Every Lie algebra of a compact Lie group can be decomposed into its center and semi-simple ideals. Thus, $\mathfrak{h} = \mathfrak{h}^{ss} + \mathfrak{Z}(\mathfrak{h})$. Denote $H^{ss} = \exp(\mathfrak{h}^{ss})$ as the semi-simple part of H^o.

Proof of 1: Let G_i be a sequence of closed subgroups of a compact group G. Define G_∞ as in 1a.

To show that G_∞ is a group, first notice that multiplication and $g \to g^{-1}$ are continuous in G. So, given $g_\infty^1, g_\infty^2 \in G_\infty$, define $g_\infty^1 g_\infty^2$ by the limit of the sequence $g_i^1 g_i^2$, where $g_i^1 \to g_\infty^1$ and $g_i^2 \to g_\infty^2$. To show that G_∞ has inverses, let $g_\infty \in G_\infty$. Let g_∞^{-1} be the limit of the sequence g_i^{-1}, where $g_i \to g_\infty$ and g_i^{-1} is the inverse of g_i in G_i. It remains to show that G_∞ is closed, so let g_∞^n be a sequence in G_∞ converging to $g_\infty^\infty \in G$. We must show $g_\infty^\infty \in G_\infty$. For each g_∞^n, find a convergent sequence of g_i^n converging to g_∞^n, which we know exists by the definition of G_∞. For each n, g_i^n converges to g_∞^n, and so we can choose $i(n)$, so that $d(g_m^n, g_\infty^n) < \frac{1}{n}$ for all $m \geq i(n)$. We will now show that $g_{i(n)}^n \to g_\infty^\infty$. Let $\epsilon > 0$. Fix n_o large enough so that $\epsilon - \frac{1}{n_o} > 0$. Now choose n_1 so that for all $n \geq n_1$, $d(g_\infty^\infty, g_\infty^n) < \epsilon - \frac{1}{n_o}$, as $g_\infty^n \to g_\infty^\infty$. Let N be the maximum of n_o and n_1. Given $n \geq N$, we see that $d(g_\infty^\infty, g_{i(n)}^n) \leq d(g_\infty^\infty, g_\infty^n) + d(g_\infty^n, g_{i(n)}^n) \leq \epsilon - \frac{1}{n_o} + \frac{1}{n} \leq \epsilon$, as desired. Now, G_∞ is a closed group in G, and so we know it is compact.

Notice that at this point, G_∞ might only consist of id $\in G$, since there may not be any convergent sequences g_i. We will next show that by taking a subsequence, we can assume that G_∞ has Property 1c, while also proving Property 1b. Property 1c implies that G_∞ is non-trivial since a Lie group does not contain any subgroup within a small neighborhood of the identity. Let U be an open neighborhood of G_∞. Assume for contradiction that G_∞ does not have Property 1c. Then $\forall i_o, \exists i \geq i_o$ so that $G_i \not\subset U$. Choose a subsequence so that $G_i \not\subset U$ and choose $g_i \in G_i$ so that $g_i \notin U$. Now, a subsequence of g_i must converge to $g_\infty^1 \in G$ outside of U or on the boundary of U. We know that $g_\infty^1 \notin G_\infty$, since we assumed $g_i \notin U$. Restrict the groups to those i's which are in the subsequence converging to g_∞^1. Form G_∞^1 corresponding to this restricted subsequence. From above, we know that G_∞^1 is a compact group. To show that $G_\infty \subset G_\infty^1$, let $g_\infty \in G_\infty$. Find a sequence $g_i \to g_\infty$. Look at the subsequence of g_i consisting only of the i's we used to form G_∞^1. This subsequence must also converge to g_∞, which is thus in G_∞^1, by definition. Recall that g_∞^1 is not in G_∞, but it is in G_∞^1, and so G_∞ is a proper subgroup of G_∞^1. If G_∞^1 does not satisfy Property 1c, then repeat the process. If this process continues on indefinitely, then we obtain a sequence of proper subgroups $G_\infty \subset G_\infty^1 \subset G_\infty^2 \subset \ldots G_\infty^n \subset \ldots$ which are compact and contained in G.

Now $(G^n)^o$, the identity component of G^n_∞, must also form a sequence of subgroups all contained in G. Look at the the Lie algebras $\mathfrak{g}^n$ of $(G^n)^o$. They must form a sequence of subalgebras. These are all in $\mathfrak{g}$, the Lie algebra of G, which is finite dimensional, so eventually they must all have the same dimension. Now $(G^n)^o$ form a connected sequence of subgroups, and eventually they must all have the same Lie algebra dimension. Hence, they must eventually be the same Lie group. Without loss of generality, we can assume that we have a sequence of proper subgroups $G_\infty \subset G^1_\infty \subset G^2_\infty \subset \ldots G^n_\infty \subset \ldots$ which are compact, have the same dimension and have the same identity component G_o.

Notice that when we choose a subsequence, then the limiting group can only get larger. Also, among all possible limiting groups, there is a largest possible dimension, but in general, one can achieve different limiting groups by choosing different subsequences. We will argue that there exists a limiting group H_∞, of largest possible dimension, such that no matter what further subsequence one chooses, the limiting group does not get any larger. If not, one gets a sequence of increasing limiting groups $H^1_\infty, H^2_\infty, ...$ (each one coming from a decreasing choice of a subsequences) all of which have the same dimension and hence the same id component but more and more components. Now one can choose a diagonal subsequence of the choice of subsequences, and the corresponding limiting group will contain all H^i_∞ and hence have infinitely many components. But this cannot be since the limiting group is a closed subgroup of G, a compact Lie group, which always has only finitely many components. Hence, there exists a limiting group H_∞ which satisfies Property 1b. In addition, since the negation of Property 1c above resulted in the formation of larger limiting groups by taking subsequences, H_∞ must also satisfy Property 1c.

Relabel G_i to the restricted subsequence of groups used to converge to H_∞ as above and in Definition 1a. Relabel so that $G_\infty = H_\infty$. Now, we know that $G_i \to G_\infty$. $\square$

Proof of 2: We can now apply a Theorem of Montgomery and Zippin [28] which says if G is a Lie group and K is a compact subgroup of G, then there exists in G an open set U containing K with the property that for each subgroup H of G lying in U, there is an element g of G such that $g^{-1}Hg \subset K$. Applying this to our case, let G be G and let K be G_∞, as above. Choose U as in Montgomery-Zippin. Choose i_o as in Property 1c. Relabel our G_i so that they begin at i_o, with $G_1 = G_{i_o}$,... We have restricted to the tail of our original group sequence, so this will not change our definition of G_∞ as in Definition 1a. Apply Montgomery-Zippin to choose $g(i) \in G$ so that $g(i)G_ig(i)^{-1} \subset G_\infty$. Look at $g(i)$. This is a sequence in the compact Lie group G, so there is a subsequence which converges. Restrict the i's further so that $g(i) \to h \in G$. To show that $h \in N_G(G_\infty)$, notice that for any $g_\infty \in G_\infty$, we can find $g_i \to g_\infty$. Restrict the g_i's to the i's we are now using. Then g_i still converges to g_∞, since we have taken a subsequence of a converging sequence. Hence, $g(i)g_ig(i)^{-1} \in G_\infty$, since $g(i)G_ig(i)^{-1} \subset G_\infty$. Taking limits, we see that $hg_\infty h^{-1} \in G_\infty$, since G_∞ is closed, and so $h \in N_G(G_\infty)$. We claim that $g(i)G_ig(i)^{-1} \to G_\infty$ is also true. Given $g_\infty \in G_\infty$, we will show that there is some conjugate sequence converging to g_∞. Look at $h^{-1}g_\infty h$, which is an element of G_∞ since $h \in N_G(G_\infty)$, so call it g'_∞. Hence, we can choose

$g_i \in G_i$ so that $g_i \to g'_\infty$, since $G_i \to G_\infty$. Restrict this sequence to the i's we now have. Then, $g(i)g_ig(i)^{-1} \to hg'_\infty h^{-1} = h(h^{-1}g_\infty h)h^{-1} = g_\infty$. Hence $g(i)G_ig(i)^{-1} \to G_\infty$. Since they are already chosen as subgroups of G_∞, then we know that $g(i)G_ig(i)^{-1} \hookrightarrow G_\infty$, as desired. □

Remark 4.6. If $G_i \hookrightarrow G_\infty$, where G_∞ is a closed group, then if we restrict the sequence, $G_{i_k} \hookrightarrow G_\infty$.

Lemma 4.7. *Let $G_i \hookrightarrow G_\infty$. Then there is a subsequence and a connected normal compact subgroup K of G^o_∞ so that $G^o_i \to K$. There exists a further subsequence so that G_i is conjugate to G'_i within G_∞, where $G'_i \hookrightarrow G_\infty$ and $(G'_i)^o \hookrightarrow K$.*

Proof of Lemma 4.7: Assume that $G_i \hookrightarrow G_\infty$. Now G^o_i are a sequence of closed subgroups of G, so apply the proof of part a) from Lemma 4.5 to obtain a group K so that $G^o_i \to K$. To see that K is normal in G_∞, let $g_\infty \in G_\infty$ and $k \in K$. Choose $g_i \in G_i$ so that $g_i \to g_\infty$, as in the definition of G_∞. Choose $g^o_i \in G^o_i$, as in the definition of K, so that $g^o_i \to k$. Notice that $G^o_i \lhd G_i$ and so $g_ig^o_ig_i^{-1} \subset G^o_i$. By taking limits, we see that $g_\infty k g_\infty^{-1} \subset K$. Therefore, $K \lhd G_\infty$.

To show that by taking a subsequence, G_i is conjugate within G_∞ to G'_i with $G'^o_i \hookrightarrow K$, notice that G_∞ is a compact subgroup of G, and so it is a Lie group. In addition, $G^o_i \subset G_\infty$ since $G_i \hookrightarrow G_\infty$. We know that $G^o_i \to K$, so apply Lemma 4.5 to $G^o_i \to K$, within the Lie group G_∞. Therefore, we can find $g(i) \in G_\infty$ so that $g(i)G^o_ig(i)^{-1} \hookrightarrow K$.

To show that K is a connected subgroup of G^o_∞, notice that conjugation preserves components, so the conjugates of G^o_i are connected. Since they are subgroups of K, they must be contained within K^o. Yet, the components of K are separated, so the conjugates must converge within K^o. Hence $K = K^o$. In addition, $G^o_i \subset G^o_\infty$, since $G_i \subset G_\infty$. Then $K \subset G^o_\infty$ since G^o_∞ is closed. Since $G^o_\infty \subset G_\infty$ and $K \lhd G_\infty$, we see that $K \lhd G^o_\infty$, as desired.

To show that $g(i)G_ig(i)^{-1} \hookrightarrow G_\infty$, notice that $g(i)G_ig(i)^{-1} \subset G_\infty$ since $G_i \subset G_\infty$, by definition of $G_i \hookrightarrow G_\infty$, and $g(i) \in G_\infty$ as above. To show that $g(i)G_ig(i)^{-1} \to G_\infty$, notice that even though we have restricted the i's used in the definition of G_∞, we still have $G_i \to G_\infty$ by the remark at the end of the proof of Lemma 4.5. Thus, by a similar argument in the proof of Lemma 4.5b, we see that $g(i)G_ig(i)^{-1} \to G_\infty$. □

Lemma 4.8. [3] *There are only finitely many semi-simple subalgebras of a compact semi-simple Lie algebra, up to conjugacy.*

Proof of Lemma 4.8: Let A be the set of $k-$dimensional subalgebras of a compact semi-simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be semi-simple, and assume that the dimension of $\mathfrak{h}$ is k. Let $B_\mathfrak{h}$ be the set of all subalgebras in A conjugate to $\mathfrak{h}$. We will show that $B_\mathfrak{h}$ is open and closed in A and hence a component of A.

To show that $B_\mathfrak{h}$ is open, let $\mathfrak{h}_1 \in B_\mathfrak{h}$. We'll show that there is a neighborhood U of $\mathfrak{h}_1$ so that $\mathfrak{h}_2 \subset U \Longrightarrow \mathfrak{h}_2$ is conjugate to $\mathfrak{h}_1$. Let H_1 and H_2 be the corresponding connected subgroups of the corresponding Lie group G. Notice that $Q_i = -B_{\mathfrak{g}_{|\mathfrak{h}_i}}$ is invariant under $\mathrm{Ad}(H_i)$ and so it extends to a bi-invariant metric on H_i, which we again call Q_i. We will now show that the diameter of H_i with bi-invariant metric Q_i is bounded independently of i.

If Q is a bi-invariant metric on G, then $\mathrm{Ric}_Q(x,y) = -\frac{1}{4}B_{\mathfrak{g}}$. To see this, note that by [13, page 103] $R(x,y)z = \frac{1}{4}[[x,y],z]$ for any Lie group with a bi-invariant metric. Therefore,

$$\begin{aligned} \mathrm{Ric}_(x,y) &= \mathrm{tr}(z \to R(x,z)y) \\ &= \mathrm{tr}(z \to \tfrac{1}{4}[[x,z],y] \\ &= \mathrm{tr}(z \to -\tfrac{1}{4}[[z,x],y] \\ &= \mathrm{tr}(z \to -\tfrac{1}{4}[y,[x,z]] \\ &= -\tfrac{1}{4}\mathrm{tr}\ \mathrm{ad}y \circ \mathrm{ad}x \\ &= -\tfrac{1}{4}B(y,x) \\ &= -\tfrac{1}{4}B(x,y), \end{aligned}$$

as desired.

Notice that $B_{\mathfrak{h}_i} = \lambda_i B_{\mathfrak{g}|_{\mathfrak{h}_i}}$ for some λ_i, and so

$$\mathrm{Ric}_{Q_i} = -\frac{1}{4}B_{\mathfrak{h}_i} = -\frac{1}{4}\lambda_i B_{\mathfrak{g}|_{\mathfrak{h}_i}} = \frac{1}{4}\lambda_i Q_i.$$

Applying Bonnet-Myers, we see that $\mathsf{diam}(\mathsf{H_i^o}, \mathsf{Q_i}) \leq \frac{2\pi}{\sqrt{\lambda_i}}$. Now, the exponential map maps $\{X \in \mathfrak{g} | Q_i(X,X) \leq \frac{2\pi}{\sqrt{\lambda_i}}\}$ onto H_i. This implies that if $\mathfrak{h}_1$ is close to $\mathfrak{h}_2$, then H_1 is close to H_2. We apply Montgomery-Zippin to H_1 and H_2 within the compact Lie group G to obtain conjugacy of H_1 and H_2. Hence we have conjugacy of the subalgebras via the exponential map.

To show that $B_{\mathfrak{h}}$ is closed, let $\mathfrak{h}_i$ be an infinite sequence of subalgebras in $B_{\mathfrak{h}}$. Now, $\mathfrak{h}_i = \mathrm{Ad}(g_i)\mathfrak{h}$, for $g_i \in G$, where $g_i \to g_\infty$ for some $g_\infty \in G$ by compactness of G. Hence, $\mathfrak{h}_i \to \mathrm{Ad}(g_\infty)\mathfrak{h} \in B_{\mathfrak{h}}$, as desired.

To show that $B_{\mathfrak{h}}$ is closed, let $\mathfrak{h}_1 \in B_{\mathfrak{h}}$.

Now $B_{\mathfrak{h}}$ is a component of A, which has only finitely many components by compactness, and so we obtain the desired result. □

Lemma 4.9. *Let $G_i \hookrightarrow G_\infty$ and $G_i^o \hookrightarrow K$. Then there exists a subsequence so that the Lie algebras of G_i^o converge as subspaces to a subalgebra of the Lie algebra of K.*

a) If the Lie algebra of K is non-trivial and semi-simple, then we can find a subsequence so that $G_i^o = K$.

b) If G_i^o is semi-simple, then K is semi-simple.

c) If K is not abelian or semi-simple, then we can find a subsequence, up to conjugacy, such that the semi-simple part of G_i^o is equal to the semi-simple part of K and $Z(G_i^o)^o \subset Z(K)$.

Proof of Lemma 4.9: We know that K is connected, as in Lemma 4.7, so consider the Lie algebras $\mathfrak{g}_i$ and $\mathfrak{K}$, all inside of the vector space V. We will show that $\mathfrak{g}_i$ converge as subspaces to a subalgebra of $\mathfrak{K}$. Since $G_i^o \subset K$, we know that $\mathfrak{g}_i$ is a subalgebra of $\mathfrak{K}$. We can assume that the $\mathfrak{g}_i$ all have the same dimension m. The set of all m dimensional subspaces in V is compact, so choose a converging subsequence and rename it $\mathfrak{g}_i$. Now $\mathfrak{g}_i$ converge to a subspace, call it $\mathfrak{s}$ of dimension m. To show that $\mathfrak{s}$ is an algebra, let $s_1, s_2 \in \mathfrak{s}$. Pick $x_i, y_i \in \mathfrak{g}_i$ so that $x_i \to s_1$ and $y_i \to s_2$. Now $[x_i, y_i] \in \mathfrak{g}_i \subset \mathfrak{K}$. Taking limits, we see that $[s_1, s_2] \in \mathfrak{s}$, as desired. and $\mathfrak{s}$ is a subalgebra of $\mathfrak{K}$. Define $S = \exp \mathfrak{s}$. Then S is a connected subgroup of K.

Proof of a): Assume that $\mathfrak{K}$ is semi-simple. If $\mathfrak{g}_i$ is not semi-simple for arbitrarily large i, then it must have a non-trivial center. Look on the group level and restrict to these i's. By Lemma 4.5, we can assume that $Z(G_i^o)^o \to L$. We still know that $G_i^o \hookrightarrow K$ by the remark at the end of the proof of Lemma 4.5. Since elements in the center will commute with all other group elements in each G_i^o, then in the limit, these elements will commute with all other group elements in K and hence converge to elements in $Z(K)$. Therefore, $L \subset Z(K)$. In addition, by Lemma 4.5, we can assume that $Z(G_i^o)^o$ are conjugate to a subgroup of $Z(K)$. Yet, we assumed that $\mathfrak{K}$ was simple, and we know that K is connected. Hence, the center of K must be finite. Hence $Z(G_i^o)^o$ are conjugate to subgroups of a connected finite group and therefore must be trivial. Hence, $\mathfrak{Z}(\mathfrak{g}_i)$ must be trivial, a contradiction to our assumption.

Therefore, $\mathfrak{g}_i$ must be semi-simple. We have already assumed that the $\mathfrak{g}_i$ all have the same dimension m. Look at $\mathfrak{g}_i = \mathfrak{g}_i^1 + \mathfrak{g}_i^2 + ... + \mathfrak{g}_i^q$, where each factor is a simple ideal. By restricting to a subsequence, we can assume that each $\mathfrak{g}_i$ has exactly q simple ideals. We know that $B_{\mathfrak{g}_i|_{\mathfrak{g}_i^p}} = \lambda_i^p B_{\mathfrak{K}|_{\mathfrak{g}_i^p}}$ for each p. The $\mathfrak{g}_i^p$ are all subalgebras of $\mathfrak{K}$, which is also semi-simple. Since they are simple, they are certainly semi-simple. There are only finitely many semi-simple subalgebras of a semi-simple Lie algebra, up to conjugacy, by Lemma 4.8. Since the Killing form is preserved under conjugacy, then there are only finitely many values for λ_i^p. They are each non-zero since $\mathfrak{g}_i^p$ is simple, so choose the smallest λ_i and call it λ. We know that λ is not zero.

Notice that $Q_i = -B_{\mathfrak{K}|_{\mathfrak{g}_i}}$ is invariant under $\mathrm{Ad}(G_i^o)$ and so it extends to a bi-invariant metric on G_i^o, which we again call Q_i. We will now show that the diameter of G_i^o with bi-invariant metric Q_i is bounded by a constant independent of i.

We know that

$$\mathrm{Ric}_{Q_i} = -\frac{1}{4}B_{\mathfrak{g}_i} \geq -\frac{1}{4}\lambda B_{\mathfrak{K}|_{\mathfrak{g}_i}} = \frac{1}{4}\lambda Q_i.$$

Applying Bonnet-Myers, we see that $\mathsf{diam}(\mathsf{G_i^o}, \mathsf{Q_i}) \leq \frac{2\pi}{\sqrt{\lambda}}$.

Let $g_\infty^o \in K$ and choose $g_i^o \in G_i^o$ with $\lim g_i^o = g_\infty^o$. Choose $x_i \in \mathfrak{g}_\mathsf{i}$ with $|x_i| \leq \frac{2\pi}{\sqrt{\lambda}}$ so that $g_i^o = \exp x_i$. Hence, x_i will converge to $x_\infty \in \mathfrak{s}$. Since exp is continuous, we know that $g_\infty^o = \exp x_\infty \in S$. Hence $K = S$. Now the connected group G_i^o are subgroups of K with the same Lie algebra dimension, so they must be the same Lie group. Hence, $G_i^o = K$.□

Proof of b): Assume that G_i^o is semi-simple. Assume for contradiction that K is not semi-simple. Then $\mathfrak{K} = \mathfrak{K}^{ss} + \mathfrak{Z}(\mathfrak{K})$.

Look at the projection $\pi : \mathfrak{K}^{ss} \oplus \mathfrak{Z}(\mathfrak{K}) \to \mathfrak{Z}(\mathfrak{K})$, which is a homomorphism of Lie algebras. Now $\pi_{|\mathfrak{g}_i} : \mathfrak{g}_i \to \mathfrak{Z}(\mathfrak{K})$ is also a homomorphism, and so $\mathfrak{a}_i = \ker(\pi_{|\mathfrak{g}_i})$, the kernel, is an ideal in $\mathfrak{g}_i$. In addition, $\mathfrak{g}_i$ is semi-simple, and so there exists an ideal $\mathfrak{b}_i$ so that $\mathfrak{g}_i = \mathfrak{a}_i \oplus \mathfrak{b}_i$. We know that $\mathfrak{b}_i$ is semi-simple since $\mathfrak{g}_i$ is semi-simple.

Then $\pi_{|\mathfrak{g}_i}$: $\mathfrak{g}_i/\ker(\pi_{|\mathfrak{g}_i}) \to \mathfrak{Z}(\mathfrak{K})$, and so $\pi_{|\mathfrak{g}_i}$: $\mathfrak{b}_i \to \mathfrak{Z}(\mathfrak{K})$ is one-to-one. Therefore $\mathfrak{b}_i$ is isomorphic to a subalgebra of $\mathfrak{Z}(\mathfrak{K})$. We have arrived at a contradiction unless $\mathfrak{b}_i = 0$, since $\mathfrak{b}_i$ is semi-simple. Hence $\mathfrak{g}_i \subset \mathfrak{K}^{ss}$. Now, repeat the proof of a) to see that $G_i^o = K^{ss}$. Hence $K^{ss} = K$, by definition of convergence of G_i^o to K. Thus, K is semi-simple, as desired.□

Proof of c): Assume that K is not abelian. As in the proof of a), we know that there exists a subsequence so that $Z(G_i^o)^o \to L \subset Z(K)$. Up to conjugacy within K and by taking a further subsequence, we may assume that $Z(G_i^o)^o \hookrightarrow L \subset Z(K)$, $G_i^o \hookrightarrow K$, and $G_i \hookrightarrow G_\infty$. This holds by arguments similar to Lemma 4.7 using the fact that $Z(kG_i^o k^{-1})^o = kZ(G_i^o)^o k^{-1}$, for $k \in K$, which holds as conjugation preserves components and the center.

Notice that if G_i were abelian for arbitrarily large i, then restricting to those groups, $G_i \hookrightarrow K$, and so K would have to be abelian, a contradiction. Hence, we know that there is an i_o so that $\mathfrak{g}_i^{ss}$ is non-trivial for $i \geq i_o$. Restricting to these i's, we have that $(G_i^o)^{ss} \neq 0$.

Define N as the compact group so that $(G_i^o)^{ss} \to N$. Up to conjugacy within K and by taking a further subsequence, we may assume that $(G_i^o)^{ss} \hookrightarrow N$, as conjugation preserves the semi-simple factor. Also, $G_i^o \hookrightarrow K$, and $G_i \hookrightarrow G_\infty$, as before. In addition, this does not affect $Z(G_i^o)^o \hookrightarrow L$. Notice that $Z(G_i^o)^o \hookrightarrow L \subset Z(K)$ implies $Z(G_i^o)^o \subset Z(K)$ and so conjugation by an element of K is the identity on $Z(G_i^o)^o$. Hence, we still have that $Z(G_i^o)^o \hookrightarrow L \subset Z(K)$.

Apply part b) to show that N is semi-simple. Now $N = (G_i^o)^{ss}$ by part a).

We will show that we can write G_i^o as $N \cdot Z(G_i^o)^o$, so let $g_i^o \in G_i^o$. First notice that both $N = (G_i^o)^{ss}$ and $Z(G_i^o)^o$ are connected subgroups of G_i^o. Look on the Lie algebra level. Then $\mathfrak{g}_i = (\mathfrak{g}_i^o)^{ss} + \mathfrak{Z}(\mathfrak{g}_i^o)$. Since G_i^o is compact, we can find $x \in \mathfrak{g}_i$, $x_{ss} \in (\mathfrak{g}_i^o)^{ss}$, and $x_\mathfrak{z} \in \mathfrak{Z}(\mathfrak{g}_i^o)$ so that $g_i^o = \exp(tx) = \exp(tx_{ss} + tx_\mathfrak{z})$, where $x = x_{ss} + x_\mathfrak{z}$. Since $[x_{ss}, x_\mathfrak{z}] = 0$, as $x_\mathfrak{z} \in \mathfrak{Z}(\mathfrak{g}_i^o)$, we know that $g_i^o = \exp(tx_{ss} + tx_\mathfrak{z}) = \exp(tx_{ss}) \cdot \exp(tx_\mathfrak{z})$. Thus, $g_i^o = n_i \cdot z_i$, for some $n_i \cdot z_i \in N \cdot Z(G_i^o)^o$, as desired.

To show that $(G_i^o)^{ss} = K^{ss}$, let $g_i^o \in G_i^o$. Then $g_i = n_i \cdot z_i \in N \cdot Z(G_i^o)^o$. Taking limits, we see that $G_i^o \to N \cdot Z(K)$, since $Z(G_i^o)^o \hookrightarrow Z(K)$. Yet $G_i^o \hookrightarrow K = K^{ss} \cdot Z(K)$. Hence, $N = K^{ss}$, and so $(G_i^o)^{ss} = K^{ss}$, as $N = (G_i^o)^{ss}$. □

Lemma 4.10. *If G is transitive on M, a connected manifold, then G^o is also transitive on M.*

Proof of Lemma 4.10: Let G^o be the connected component of the identity. By a standard argument, we will show that any orbit is both open and closed. Notice that G_∞^o is compact and hence the orbit is compact and hence closed. To show that an orbit is open, pick a point p in the orbit of G_∞^o. In M, we can find a neighborhood around this point which stays within the orbit. Intersect this neighborhood with a neighborhood of the identity. Since p is in the orbit of G^o_∞, we know this is non-empty. Hence any orbit is open. Since M is connected, we know that the orbit must be everything or empty. We know the orbit is not empty, so it is everything. Therefore, S^n/G^o_∞ is transitive.□

Lemma 4.11. *The following is the classification of connected transitive orthogonal subgroups and the spheres on which they act:*[2]

1)$SO(n+1)$ on S^n
2)$U(n) \subset SO(2n)$ on S^{2n-1}
3)$SU(n) \subset SO(2n)$ on S^{2n-1}
4)$Sp(n) \times Sp(1) \subset SO(4n)$ on S^{4n-1}
5)$Sp(n) \times U(1) \subset Sp(n) \times Sp(1)$ on S^{4n-1}
6)$Sp(n) \subset Sp(n) \times Sp(1) \subset SO(4n)$ on S^{4n-1}
7)$\mathrm{Spin}(9) \subset SO(16)$ on $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$
8)$\mathrm{Spin}(7) \subset SO(8)$ on $S^7 = \mathrm{Spin}(7)/G_2$
9)$G_2 \subset SO(7)$ on $S^6 = G_2/SU(3)$

Theorem 4.3 *If G is a non-transitive group, then there exists ϵ, depending only on n, so that* $\mathsf{diam}(\mathsf{S^n}/\mathsf{G}) \geq \epsilon(\mathsf{n})$.

Proof of Theorem 4.3: Fix n, and assume for contradiction that we have a sequence of closed, non-transitive $G_i \subset O(n+1)$ so that $\mathsf{diam}(\mathsf{S^n}/\mathsf{G_i}) \to 0$.

We know that conjugation does not change the diameter of the resulting quotient. Therefore, changing the sequence via restriction and conjugation, will not change that $\mathsf{diam}(\mathsf{S^n}/\mathsf{G_i}) \to 0$. So, without loss of generality, we may define G_∞ as in Lemma 4.5 and we may assume that $G_i \hookrightarrow G_\infty$ and $G^o_i \hookrightarrow K$ by Lemmas 4.5 and 4.7.

Now $\mathsf{diam}(\mathsf{S^n}/\mathsf{G_\infty}) = 0$ since a group has smaller resulting diameter when compared with the resulting diameter of its subgroups. Since G_∞ is closed, then G_∞ must be transitive on S^n, as desired.

If $K = \mathrm{Id}$, then $G^o_i = \mathrm{Id}$. Hence, G_i is finite. Yet, the diameter of $S^n/G_i \to 0$. This contradicts Theorem 3.14 which produces a lower bound, $\epsilon(n)$, on the diameter resulting from finite groups . Hence $K \neq \mathrm{Id}$ and $G^o_i \neq \mathrm{Id}$.

Case 1): G^o_∞ is simple

We know that K is connected and normal in G^o_∞. In addition, G^o_∞ is simple and $K \neq \mathrm{Id}$. Hence, we know that $K = G^o_\infty$. Hence K is transitive and simple. Apply Lemma 4.9 to see that $G^o_i = K$. Since K is transitive, we see that G^o_i is transitive, and so we have arrived at a contradiction.

Case 2): G^o_∞ is not simple

Since G^o_∞ is not simple, 2), 5) and 4) in Lemma 4.11 are the only groups which remain to be examined.

Converting the notation in Lemma 4.11 to our notation in $O(n+1)$ acting on S^n, we see that G^o_∞ is not simple only when $n+1$ is even. Let $n+1$ be even. Let $j = \frac{n+1}{2}$.

Then G^o_∞ is one of the following,

$$\begin{aligned} &a) U(j) \subset SO(n+1),\\ &b) Sp(\tfrac{j}{2}) \times Sp(1) \subset SO(n+1), \text{if j is even}\\ &c) Sp(\tfrac{j}{2}) \times U(1) \subset SO(n+1), \text{if j is even}\end{aligned}$$

where b) and c) only occur if $n+1$ is divisible by 4.

We will first examine these actions. In a), write $\mathbb{R}^{n+1} = \mathbb{C}^j$. Then $A \in U(j) : v \to Av$, and $z \in U(1) = Z(U(j)) : v \to zv$. In b), examine $\mathbb{R}^{n+1} = \mathbb{H}^{\frac{j}{2}}$. Then $A \in Sp(\frac{j}{2})$ and $q \in Sp(1)$ acts via $v \to A(v)q$. In c), $A \in Sp(\frac{j}{2})$ and $z \in U(1) \subset Sp(1)$ acts via $v \to A(v)z$.

We know that, K is connected, non-trivial and a normal subgroup in G^o_∞, and that G^o_i are subgroups of K which converge to K.

We will now more closely examine the actions. In a), the only proper, connected, normal subgroups are $U(1)$, the Hopf action, and $SU(j)$, which acts transitively. In b), the only proper, connected, normal subgroups are $Sp(\frac{j}{2})$, which acts transitively, and $Sp(1)$. In c), the only proper, connected normal subgroups are $Sp(\frac{j}{2})$, which acts transitively, and $U(1)$.

Hence, we see that K must be G^o_∞ itself, or $SU(j), U(1), Sp(\frac{j}{2})$, or $Sp(1)$. If $K = SU(j)$ or $Sp(\frac{j}{2})$, then K is simple and transitive, so apply arguments in Case 1) to obtain a contradiction. We will examine the remaining groups to obtain a contradication in each case.

Case 2A): $K = U(1)$

If we are in a), then the $U(1)$ action on $\mathbb{R}^{n+1}$ is the Hopf action. To further examine the $U(1)$ action in c), look at $\mathbb{C}^{2m} = \mathbb{H}^m$ via $(a,b) \to a + jb = v$. To show that the $U(1)$ action is also the Hopf action, notice that
$z(a,b) = (za, zb) = (az, bz) \to az + j(bz) = (a + jb)z = vz$.

We know that $G^o_i \neq \mathrm{Id}$ is connected and a subgroup of $K = U(1)$. Hence, G^o_i is $U(1)$. Now,

$$S^n/G_i = \frac{S^n/G^o_i}{G_i/G^o_i} = \frac{S^n/U(1)}{G_i/U(1)} = \frac{\mathbb{C}P^{j-1}}{G_i/U(1)}.$$

Using the submersion metric, we see that $G_i/U(1)$ is in the isometry group of $\mathbb{C}P^j$, since $G_i \lhd U(1)$ and so orbits get mapped to orbits. We know that $G_i/G^o_i = G_i/U(1)$ is finite. Therefore $G_i/U(1)$ is a finite subgroup of the isometry group of $\mathbb{C}P^n$. Apply Corollary 3.14.1 to obtain a lower bound on the diameter of $\frac{\mathbb{C}P^{j-1}}{G_i/U(1)}$. We have arrived at a contradiction to the assumption that the diameter of S^n/G_i approaches 0.

Case 2B): $K = Sp(1)$

We know that $G^o_i \neq \mathrm{Id}$ is connected and a subgroup of $K = Sp(1)$. But, the only connected subgroups of $Sp(1)$ are great circles in S^3, which cannot converge to

$S^3 = Sp(1)$. Hence, G_i^o is $Sp(1)$. Also,

$$S^n/G_i = \frac{S^n/G_i^o}{G_i/G_i^o} = \frac{S^n/Sp(1)}{G_i/Sp(1)} = \frac{\mathbb{H}P^{\frac{j}{2}-1}}{G_i/Sp(1)}.$$

The contradiction is obtained in a similar fashion to Case 3A), using Corollary 3.14.1 to obtain a lower bound on the diameter of $\frac{\mathbb{H}P^{\frac{j}{2}-1}}{G_i/Sp(1)}$.

Case 2C): $K = G_\infty^o$

K is transitive. Even though K is not simple, we will still be able to apply arguments which resemble those in Case 1).

Case 2C1): $K = Sp(\frac{j}{2}) \times Sp(1)$

Now K is semi-simple, so apply part a) of Lemma 4.9 to restrict the sequence so that $G_i^o = K$. Since K is transitive, we see that G_i^o is transitive, and so we have arrived at a contradiction.

Case 2C2): $K = U(j) \subset SO(n+1)$ or $Sp(\frac{j}{2}) \times U(1)$

We satisfy the conditions of part c) of Lemma 4.9, since K in not abelian or semi-simple. By Lemma 4.9 we can restrict the sequence so that up to conjugacy, $(G_i^o)^{ss} = K^{ss}$. We have arrived at a contradiction since $K^{ss} = SU(j)$ or $Sp(\frac{j}{2})$, which is transitive.

There are no other cases remaining, so the theorem is proven. $\square$

Summary of Diameter Results				
Class of Groups or Actions	*Resulting Spaces*	*Lower Bound on Diameter*	*Dimension Achieved*	*Limit as $n \to \infty$*
Reducible Actions (Grove, Borzellino)		$\frac{\pi}{2}$		$\frac{\pi}{2}$
Free Actions (McGowan,91, Flach,92)	Manifolds	$\frac{\pi}{9.63}$	3	$\frac{\pi}{2}$
Dimension 2 Groups ([20],[15])	Orbifolds and Alexandrov Spaces	$\frac{\pi}{4.82}$	2	
Coxeter Groups	Coxeter Orbifolds	$\frac{\pi}{8.10}$	3	$\frac{\pi}{2}$
Finite Groups	Orbifolds	$\exists$ explicit $\epsilon(n)$	n	0
Cohomogeneity-1 Actions (Hsiang-Lawson,71)	Intervals	$\frac{\pi}{6}$	7 and 13	$\frac{\pi}{4}$
Infinite, Closed, Non-Transitive Groups	Alexandrov Spaces	$\exists\epsilon(n)$	n	

TABLE 2. Summary of Diameter Results

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