Mathematics In Depth

I will be discussing both Markov matrices and game theory in depth. These are two topics that Blackwell did a lot of research in and enjoyed learning about.

Introduction to Markov Matrices

To understand what a Markov matrix is we must first define a probability vector. “A probability vector has entries tat are non-negative numbers and add to one. An example of a probability vector would be [.5, 0, 0, .5]. Square matrices whose columns are probability vectors are called stochastic matrices or Markov matrices. An example of Markov matrices is as follows $\begin{bmatrix} 1 & .5 & 0 & 0 \\ 0 & 0 & .5 & 0 \\ 0 & .5 & 0 & 0 \\ 0 & 0 & .5 & 1 \end{bmatrix}$. We will talk later about Markov chains.

Markov matrices can be used in predicting how things will pan out in the future. A perfect example would be to try and see how populations will turn out in the future. A scenario that we might be interested in would be to see how populations migrate within a country from urban to rural settings.

Background on the Example

For example (3) in a particular country the current population has 45% living in urban areas and 55% living in rural areas. Each year 6% of the urban population moves to rural areas and 94% stay in the urban areas. Also each year 9% of the rural population moves to the urban area and 91% stay in the rural areas. We are now concerned with what will happen to the population in the future. A person may wonder if eventually everyone will live in urban areas or if the trend is to live in the rural areas. To figure this out we must form a matrix.

Equation Formation to Matrix Formation

The easiest way to make up he matrix is to write the equations that explain how the population moves each year. We will use u_k to represent the urban population in k years and r_k to represent the rural population kin k years. The equation for the urban population is as follows. $U_{k+1}=.94u_k+.09r_k$.

This equation states that 94% of the original population plus 9% of the rural population is the new population after k years. The equation for the rural population is as follows

$R_{k+1}=.06u_k+.91r_k$.

This equation states that 6% of the urban population plus 91% of the rural population is the new population of the rural area after k years. From these tow equation we can form the following matrix $\begin{bmatrix} u_{k+1} \\ r_{k+1} \end{bmatrix} = \begin{bmatrix} .94 & .09 \\ .06 & .91 \end{bmatrix} \begin{bmatrix} u_k \\ r_k \end{bmatrix}$. We will explain why this is the correct matrix in a moment.

Matrix Multiplication

To understand dhow these matrices work we should review matrix multiplication. Using he following matrix we will show the process. $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} (a_1 * c_1) + (a_2 * d_1) \\ (b_1 * c_1) + (b_2 * d_1) \end{bmatrix}$

Why the Matrix is Correct

Looking at the previous matrix dealing with the population situation we can substitute uk for c1 and rk for d1 to get $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} u_k \\ r_k \end{bmatrix}$. Then applying the matrix multiplication we see we get the following equations.

I. $A_1*u_k+a_2+r_k=u_{k+1}$. From this equation you can see the .94 must be a_1 and .09 must be a_2.

II. $B_1*u_k+b_2*r_k=r_{k+1}$. From this equation you can see that .06 must be b_1 and .91 must be b_2.

Now we should check to see what would happen if the columns were to be reversed. If we were to reverse the columns the matrix would look like $\begin{bmatrix} u_{k+1} \\ r_{k+1} \end{bmatrix} = \begin{bmatrix} 09 & .94 \\ 91 & .06 \end{bmatrix}\begin{bmatrix} u_k \\ r_k \end{bmatrix}$. Now if we apply matrix multiplication we will get the following equation.

I. $U_{k+1}=.94*u_k+.94*r_k$ which states that 9% of the urban population plus 94% of the rural population makes the new population for the urban areas after k years. His is an incorrect statement.

This shows the importance of having the columns in the correct order. Now we must check to be sure that this matrix is a Markov matrix. We must check to see that each column of the matrix $\begin{bmatrix} 94 & .09 \\ 06 & .91 \end{bmatrix}$ adds up to one.

Solving the Problem

Now that we have established how to form the matrix we need to look into the original question of what will happen to the population in the future. If we take the Markov matrix and raise it to the number of years we are interested in then multiply it by the original distribution we will receive the new population. Like stated earlier the original population for the urban are was 45% and 55% for the rural area. The statement just explained looks like the following equation.

$$\begin{bmatrix} u_{k+1} \\ r_{k+1} \end{bmatrix} = \begin{bmatrix} 09 & 394 \\ 91 & 06 \end{bmatrix}^k \begin{bmatrix} 45 \\ 55 \end{bmatrix}$$

Now applying the formula we will look to see what happens after 10 years, 50 years, 100 years, and 150 years. We get the results by taking the Markov matrix and multiplying it by itself the number of years we are interested in. The results we get are as follows.

$$\begin{bmatrix} 5704688394 \\ 4295311606 \end{bmatrix}$$

$$\begin{bmatrix} 5999556353 \\ 4000443647 \end{bmatrix}$$

$$\begin{bmatrix} 5999999869 \\ 4000000132 \end{bmatrix}$$

$$\begin{bmatrix} 6000000000 \\ 4000000001 \end{bmatrix}$$

These matrices are what we call a Markov chain. In this example we see that the matrix is converging to $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$. This matrix is called the steady-state vector. Mot all Markov matrices will converge to a steady-state vector. The ones that do converge are considered “regular” stochastic matrices.

Background on the Game

In the paper "The Big Match" Blackwell deals with a two-person zero sum game. A game is an interaction or exchange between tow (or more) players, where each player attempts t make a choice (or "move") towards the other player in such a way that he could expect a maximum gain, depending on the other's response. Zero-sum games are games where the amount of "winnable goods' is fixed. The other player therefore loss whatever the other player gains: the sum of gained (positive) and lost (negative) is zero.

How to Play the Game

In Blackwell's paper he explains the following game. "Every day player 2 chooses a number, 0 or 1, and player 1 tries to predict player 2's choice, winning a point if he is correct. This continues as long a player 1 predicts a 0. But if he ever predicts a 1, all future choices for both players are required to be the same as that day's choice; if he is wrong on that day, he wins zero every day thereafter." This is a bit complicated but in simpler terms as long as player 1 keeps choosing 0, player 2 can keep pick either 0 or 1. Then as soon as player 1 picks 1 both players must keep their choice of that day. This means that either player 1 guessed correctly and will now win everyday form no on or he guessed wrong and will now lose everyday.

Now we will go through a few situations of how the game might work out.

Example 1: Player 2- 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0…

Player 1- 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1,…

In this example we see that player 1 wins seven times before he chooses 1 and then he loses form there on out.

Example 2: Player 2- 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1,…

Player 1- 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1,…

In this example we see that player 1 wins five times before eh chooses 1 and then he wins from there on out.

Strategies for Winning

There is a 50% chance that player 1 will guess the correct choice of player 2 since he is either going to guess right or wrong. The best strategy for player 2 is tossing a fair coin every day. This strategy is the best because there is a 50% chance that player 1 will guess his choice. Unfortunately, player 1 has no optimal strategy. The only thing he might want to do is to continue picking 0 noting when he loses and wins and see if there is a probability of when it would be best for him to pick a 1. Blackwell has formed a formula to help player 1 decide when he should pick 1 or if he should never pick 1. With the following example I will talk you though this procedure. In the formula we will use an N, which is a large non-negative number. We will keep N an arbitrary number. We now need to find the excess of zeros over ones. At any given stage of the game, which is the same thing as the number of wins over losses for player 1. We find this excess because the game is still continuing so player 1 has chosen all 0's up to this point. Lets assume that player 2 has picked 99 1's and one 0 and player 1 has only chosen 0's up to this point. In this case it is 1-99=-98, which is equal to k_{100} or the excess of 1's over 0's. Then pick with the probability of $P(k_n+N)=P(-98+N)$ where $P(m)=1/(m+1)2$. The final probability is $P(-98+N)=1/((-98+N)+1)/2$. This is a very small denominator create a larger probability that if player 1 were to guess a 1 there is a greater chance that he would be correct. Now there are two factors involved when looking at this probability. For one, player 1 sees that there are 99 1's and so he would expect that if player 2 is using a fair coin then a 0 should be coming soon. Therefore he would continue picking a 0. On the other hand, we do not know if player 2 is using a coin. It is possible that he is using another strategy to pick his number. The second factor involved is that player 1 has been losing a lot and therefore you would believe that player 1 should start winning and thereof would want to stay picking 0. This formal is only a strategy for player 1 to go by. IT is important for player 1 to not pick 1 until he is certain that a 1 is coming next because if they guess a 1 and player 2 has chosen a 1 then player 1 will lose from there on out.

At the other extreme we can look at another example where player 2 chooses 99 0's and one 1. In this case k100 is 99-1=98. P(Kn+N)=P(98+N) where P9m)=1/((98+N)+1)2. This creates a large denominator making the probability a very small probability. This probability states that if player 1 were to guess a 1 there is a small chance that he would be correct. Once again there are two factors that we must consider along with this probability. One facto is that player 2 has chosen 99 0's and if he were using a coin then player 1 would expect a 1 to come along soon. Therefore player 1 would want to pick a 1. There is a possibility that player 2 is not using a coin and using some other method to pick his number. The other factor that we must take into consideration is that player 1 has been winning a lot and he would think it would make since to start losing soon therefore leading him to want to pick a 1.

References

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