IN DEPTH MATHEMATICS

One of the most well known methods of finding irreducibility of polynomials with integer coefficients is demonstrated by Eisenstein's criterion. When something is irreducible, it can't be factored into smaller polynomials with rational coefficients. For example:

$$X^2-1=0$$
$$(x+1)(x-1)$$

This is reducible because it breaks down into a smaller polynomial with rational coefficients.

$$x^2+1=0$$
$$(x+i)(x-i)=0$$

this is irreducible because when it's broken down, it leaves irrational coefficients.

Eisenstein's criterion states that if all the coefficients, except possibly the first one, are divisible by a prime "p", and the constant coefficient is not divisible by p^2, then the polynomial is irreducible. His equation is the following:

$$X^n + A_{n-1} X^{n-1} \ldots + A_o = 0$$

When you come across a complicated polynomial you can try using this method; however, this doesn't always work. It only works if the polynomial follows the rules stated above. For example:

$X^2 + 10X + 5 = 0$

When you look as this equation you notice the prime the coefficients have in common is 5. Therefore, p=5. The next thing you notice is that ($5/p^2$) does not give you a rational number when it is divided; therefore, this equation is irreducible.

$X^2 – 8X + 4 = 0$

In this equation your coefficients have the prime number 2 in common; therefore, your p=2. The next step is dividing it's square by the constant coefficient. In this case ($4/p^2$) gives us a rational number. This concludes that you cannot use Eisenstein's criterion on this equation.

$X^2 – 4X + 2 = 0$

The prime number that the coefficients have in common is 2 and when you divide ($2/2^2$) it comes out as an irrational number; therefore, by Eisenstein's criterion, it's irreducible.

It sometimes happens that the criterion is not applicable to the polynomial because it does not follow the criteria. For example:

$X^4 + 1=0$

In Eisenstein's criterion the X's follow a decreasing pattern:
$X^n + A_{n-1} X^{n-1} \ldots + A_o=0$
In this case it's
$X^n + A_o = 0$, so it's not applicable.

For every great equation there is always a trick if something doesn't work out. In this case, since $X^4 + 1= 0$ does not follow Eisenstein's criteria, we can transform it into something that'll work, for example:

$F(x)=X^4 + 1 = 0$ ⑧ $g(x) = f(x+1) = (x+1)^4 + 1 = 0$

$(x+1)*(x+1)*(x+1)*(x+1)+1=0$

$X^4+4X^3+6X^2+4X+2=0$

Now this polynomial satisfies the conditions of the Eisenstein's criterion. We find that p=2 and since $(2/2^2)$ leaves us with an irrational number, we conclude that this polynomial is irreducible.

This trick works because any factor of f(x) would be a factor of g(x) by substituting "x" by (x+1) in each factor. However, this trick doesn't always work:

$f(x)=X^3+1=0$ ⑧ $g(x)=(x+1)^3-1=0$

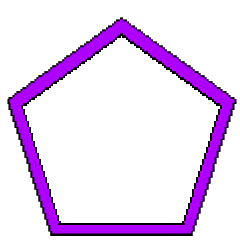
$(x+1)*(x+1)*(x+1)+1=0$

$x^3+3x^2+3x+2=0$

In this case, transforming The function into g(x) still didn’t help us solve it because you this equation still doesn’t have the criteria needed to use Eisenstein’s criterion.

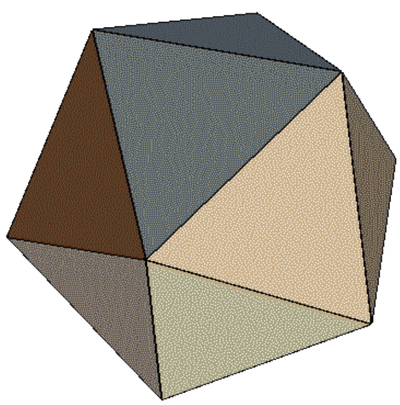
Eisentein’s criterion basically reduces the problem of factoring a difficult polynomial to a problem of factoring integers by using the coefficients of the former polynomial to see if they have a common prime divisor.

Grace Murray Hopper, instead of the open Dumas polygon, introduced the closed convex polygon, which is applied to the deduction of irreducibility criteria. This was dependent on the size and the divisibility properties of the coefficients. For the closed convex polygon, an approximate multiplication theorem holds and may be used to deduce irreducibility criteria depending on the size of the coefficients. A convex polygon is a closed figure in a plane whose angles are less than 180 degrees. For example:

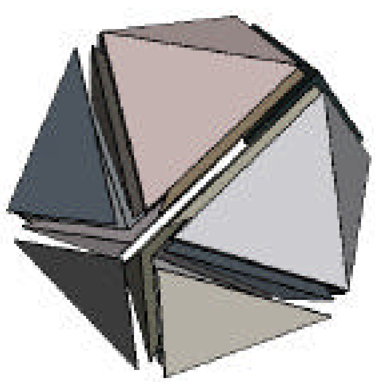


This is a convex polygon because
the angles are less than 180 degrees

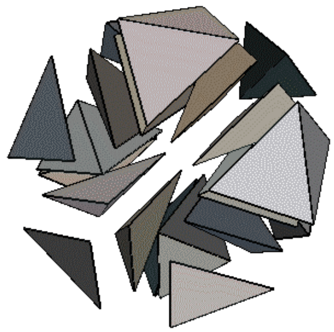
Grace Murray Hopper found a way to convert a polynomial into a convex polygon. With this conversion she found a way to decompose the polygon the way that Eisenstein broke down the polynomials.



This is an example of an icosehedron, which is going to be decomposed into a tetrahedron.



This shows the decomposition the icosahedron and how it is broken up.



This is the final step and the one that Hopper used in order to solve for irreducibility. This is all that I know about Hopper because all I had to work with was an abstract of her paper. The basis of her work was irreducibility and the process of turning then into closed convex polygons and determining their reducibility.