In Depth Math

Olga's main focus was number theory, but she was later introduced to a branch of math called matrix theory. "A matrix is a rectangular array of symbols, usually numbers, neatly arranged in columns and rows" (Math Trek, 1). Matrices come into play in a lot of math aspects. Some of these aspects are algebra, differential equations, probability, and other fields as well. Engineers and theoretical physicists use matrices as well (Math Trek, 1).

Taussky-Todd was introduced to matrix theory during WWII after taking a position at the National Physical Laboratory in London. She worked here with a group investigating flutter, which is an aerodynamic phenomenon.

> In flight, interactions between aerodynamic forces and a flexing airframe induce vibrations. When an airplane flies at a speed greater than a certain threshold, those self-excited vibrations become unstable, leading to flutter. Hence, in describing an airplane, it's

important to know what the flutter speed is before the aircraft is built and flown (Math Trek, 1).

Engineers had to use certain differential equations to estimate the flutter speed and this process led to finding the eigenvalues of a square matrix. “An eigenvalue is the scalar multiple of nonzero vectors of a given matrix” (paper on the Internet, 4). A square matrix is one in which the number of rows equals the number of columns (i.e. an mxm matrix). Eigenvalues are useful in geometry when dealing with R^3 space and vectors along a line given by $\underline{\lambda x}$. They can be determined algebraically also. Given a matrix ‘A’, you can find the eigenvalues by evaluating the determinant of (λI-A) and setting that equal to zero. The equation looks like this: $\underline{\det(\lambda I-A)=0.}$ Eigenvalues are useful in dynamical systems as well. Olga used them to help find the vibrations that interactions between aerodynamic forces and a flexing airframe induce. This was a very time consuming task. Olga found a way to reduce the amount of calculation. She wanted to refine a method for getting useful information about the eigenvalues without having to go to all the extra trouble involved in computing them exactly. She used a theorem named for a Russian mathematician called the Gerschgorin Circle Theorem (Math Trek, 1).

This theorem deals with a square matrix that has entries that can be complex numbers. "A complex number has two parts and can be written as *a*+*bi*, where *a* is the real part and *bi* is the imaginary part, with *i* representing the square root of –1" (Math Trek, 1). Each complex number has a real x-coordinate and an imaginary y-coordinate. The complex number 2+5i would be plotted as the point (2,5).

Olga began to use the theorem as a way to zero in on the eigenvalues graphically. "The theorem states that the eigenvalues of an n x n matrix A with complex entries lie in the union of closed disks, the Gerschgorin disks in the complex z plane" (paper, 4).

The Gerschgorin Circle Theorem:

Let A be an n x n matrix and R_I denote the circle in the complex plane with center a_{ij} and the radius

$$R_i = \{z \in C \text{ such that } |z - a_{ii}| \le \Sigma |a_{ij}|\}$$

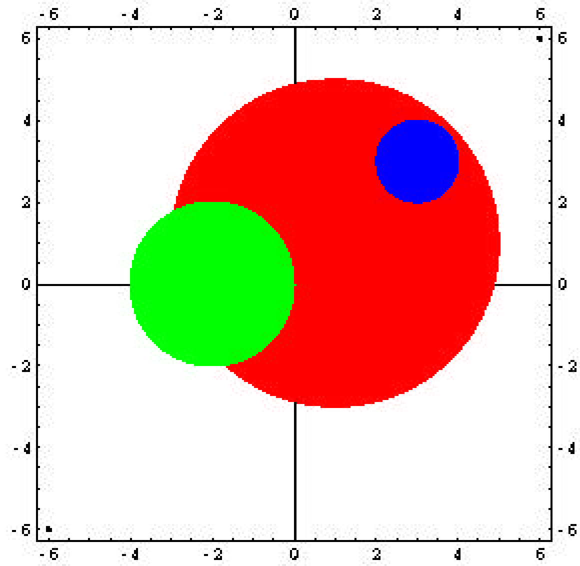
Where the sum runs j≠i from j=1 to n, and C denotes the complex plane. The eigenvalues A are contained within $R = \bigcup R_i$, where i runs from 1 to n. Moreover, the union of any k of these circles that do not intersect the remaining (n-k) contain precisely k (counting multiplicities) of the eigenvalues.

Here is an example of a square matrix with complex entries:

1+i	3	2
1	2+7i	0
0	4	-2

"All of the eigenvalues of this matrix lie under the union of certain disks, whose centers are the values along the diagonal and whose radii are the sum of the absolute values of the off-diagonal entries in a given row" (Math Trek, 2). The previous statement can be made according to the Gerschgorin Circle Theorem.

If we look at this matrix, the circle corresponding to the first row would be centered at the point (1,1) and have a radius of 5. The second circle would be centered at the point (2,7) and have a radius of 1. The third circle would have its center at (-2,0) and a radius of 4. Hence, the three eigenvalues would be complex numbers that lie somewhere in the complex plane within the areas defined by those circles.



In flutter equations, those disks had a particular pattern. This was a break for Taussky-Todd and allowed her to develop ways to make the circles smaller so that they would not overlap as much and would provide much sharper estimates of the eigenvalues (Math Trek, 2).