Mathematics

Doris Schattschneider worked on many aspects of mathematics. Symmetry and geometric models have long held a special fascination with her. Doris was interested in both geometry and art and this led naturally to the study of tiling problems and the work of the Dutch artist M.C. Escher. She has written many scholarly articles that have dealt with many subjects within mathematics. One of her articles dealt with taxicab geometry. In this discussion we will concentrate on her work in taxicab geometry.

Taxicab geometry is a field of geometry that differs greatly from the Euclidean geometry that most of us are used to. In taxicab geometry the only way that you can move between two points is to move in a horizontal or vertical straight line or turn at a right angle. Think of a taxicab driver on a city grid. The driver can only follow the streets. He must either go straight or turn left or right (at a right angle.) This is how taxicab geometry got its name. This differs from Euclidean geometry in that in Euclidean geometry you connect any two points with a straight line. Recall the distance equation from Euclidean geometry. The distance between any two points (x_1, x_2), (y_1, y_2) is given by

$$D = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}.$$

In taxicab geometry the distance equation changes. The distance between those same two points in taxicab geometry is given by

$$D_T = \text{absolute value of } (x_2 - x_1) + \text{absolute value of } (y_2 - y_1).$$

This is known as the taxicab metric. The taxicab metric was first discovered by Hermann Minkowski (1864-1909) as a special case of a metric defined in terms of an arbitrary convex set centered at the origin. Consider the two points (1,5) and (10,8). The distance between these two points in Euclidean geometry is $(90)^{1/2}$ and their distance in taxicab geometry is 12. (These distances come from the equations above.)   It is obvious that taxicab distance will always be greater than Euclidean distance. This should make perfect sense since we know that the shortest distance between two points is a straight line.

Another interesting fact about taxicab geometry is that Euclid's fourth postulate, side angle side does not hold. To show that side angle side does not hold we need to show one case where the side, angle, and side are the same but the length of the hypotenuse is different. Consider the points (0,4), (0,0) and (2,0) and the triangle they make. One side of the triangle is 4, the other 2 and the angle is 90^o. The distance of the hypotenuse is the distance between (0,4) and (2,0) which is 6 using the taxicab metric. Now consider the points (1,1), (2,0) and (3,3) and the triangle they make. The sides are again 4 and 2 and the angle between them is again 90^o. However, now the distance of the hypotenuse is the distance between (1,1) and (3,3) which is 4 using the taxicab metric. Now, we have shown that we can have the same side, angle, side and a different hypotenuse, therefore side angle side does not hold in taxicab geometry.

This new definition of distance tends to make one wonder where taxicab geometry is used. Other than the most obvious case with a taxi driver and a city, taxicab geometry can be used anytime to find distance when traveling in a straight line is not possible. Also biologists have found the metric D_T useful in the measurement of "niche overlap" and notion of ecological distance between species.

For this paper we will look at Schattschneider's paper entitled "The Taxicab Group." In the paper Schattschneider addresses the problem of "given a space S, endowed with a metric d, describe the group G of isometries with respect to the metric d." The metric d she was referring to was the taxicab metric that was explained earlier. First, an isometry is a translation, rotation, reflection or glide reflection that preserves distance between two points in the plane. To answer this question Schattschneider first made some geometric observations. The first observation is that if x and y are points in the plane, then the taxicab distance is just the length of the L shaped figure that joins them, where the sides of L are parallel to the coordinate axes. Because of this observation we can immediately rule out Euclidean isometries that map horizontal or vertical lines to lines that are not parallel to the coordinate axes; for doing this would increase taxicab distance. The only time taxicab and Euclidean distance are the same is when the points lie on a straight line. If the points were not parallel to either axis then their taxicab distance would increase because you would have to make at least one 90^{o} turn to get to the other point therefore increasing your distance. After making this observation, Schattschneider sates that there are only eight types of Euclidean isometries that preserve distance that we must consider. The eight isometries are as follows: (1) translation, (2) rotation of 180^{o}, (3) glide reflection in horizontal line, (4) glide reflection in vertical line, (5) glide reflection in line with slope –1, (6) glide reflection in line with slope 1, (7) rotation of 90^{o}, and (8) rotation of 270^{o}. The translation is just the figure itself. The rotations of 90^{o}, 180^{o}, and 270^{o} are self-explanatory. The glide reflections are reflections that contain the identity mapping and the line that are mentioned are what they are reflected about (horizontal, vertical, etc).

After observing that these eight isometries in Euclidean geometry preserve distance in taxicab geometry, Schattschneider goes on to ask the question do anymore exist? Schattschneider then goes through a rather complicated proof in which she blends geometry and algebra to arrive at her conclusion. We won't go through that proof here, however Schattschneider did indeed prove that the eight that she observed first were the only eight that existed.

References:

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