Tapia's Mathematics

Richard A. Tapia, in his paper entitled “The Weak Newton Method and Boundary Value Problems,” was working with the Euler-Langrange equation to try to find the maximum and minimum points(also called the roots). A much simplier version, which we are going to talk about, is regular Newton’s method. The idea behind this method is that we can find the roots of systems of equations by estimating the x-intercept of the tangent line. First you begin by choosing an initial point in the complex plane, and then iterate until you get better and better guesses. If we start close to the root, then Newton will converge quadratically (Hirst).

To solve a nonlinear system of equations using Newton’s method, we use the formula

$$\mathbf{x^{(k+1)} = x^{(k)} - [f'(x^{(k)})]^{-1}\, f(x^{(k)}).}$$

We can also see that this is the tangent line approximation. In Newton’s method, the x-intercept of the tangent line is the root estimate, in our case (x_k,y_k). The slope of the tangent line through these points is $f'(x)$. Now, using the equation of a line we get $y-y_k = f'(x_k)(x-x_k)$. Next, we want to solve for x, which changes the equation to $x = x^k+(y-y^k/f'(x^k))$. Because we want to find the roots of the equation which occur when the tangent line crosses the x-axis, then y must be zero. Plugging in zero for y, $f(x^k)$ for y_k, and then bringing $f'(x^k)$ up to the numerator gives us the above formula for Newton’s method (Analysis).

In order to understand the formula for Newton’s method, you must know a little more about what the equation represents. To do this, we had to write it in matrix/vector form. In the equation above, $f'(x^{(k)})$ is the Jacobian matrix or $\begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{vmatrix}$. Thus, the inverse of the Jacobian matrix is $1/wron*\begin{vmatrix} \partial f_2/\partial y & -\partial f_1/\partial y \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{vmatrix}$. Now the equation becomes

$\mathbf{x^{(k+1)} = x^{(k)}-1/wron}*\begin{vmatrix} \partial f_2/\partial y & -\partial f_1/\partial y \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{vmatrix}$. We can simplify this further by plugging in our initial points for $\mathbf{x^{(k)}}$. As a result we have $\begin{vmatrix} x^{n+1} \\ y^{n+1} \end{vmatrix} = \begin{vmatrix} x_0 \\ y_o \end{vmatrix} -1/wron*\begin{vmatrix} \partial f_2/\partial y & -\partial f_1/\partial y \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{vmatrix}*\begin{vmatrix} f_1(x_o,y_o) \\ f_2(x_o,y_o) \end{vmatrix}$.

Then after we multiply the equation out, we get $x_n = x_{n-1}-(\partial f_2/\partial x*f_1(x_{n-1},y_{n-1})-\partial f_1/\partial y*f_2(x_{n-1},y_{n-1}))/wron$, where $f_1(x_{n-1},y_{n-1})$ and $f_2(x_{n-1},y_{n-1})$ are the functions evaluated at our two points x_o and y_o (Hirst).

Now, to demonstrate Newton's method we will look at the following system of equations with the starting point (-0.5,.25).

$$eq1:3x^2+4y^2-1$$
$$eq2:y^3-8x^3-1$$

The Jacobian matrix for this system turns out to be $\begin{vmatrix} 6x & 8y \\ -24x^2 & 3y^2 \end{vmatrix}$. From this we can calculate the wron by plugging in our initial points for x and y, and then taking the determinant of the Jacobian matrix. For instance, the determinant of the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is ad-bc. Hence, the wron should become 11.4375. From here we calculate our new x and y by the following two formulas:

$$x^{(n+1)} = x^{old} - ((\partial f_2/\partial y)* eq1 - (\partial f_1/\partial y)*eq2)/wron$$
$$y^{(n+1)} = y^{old} - ((\partial f_1/\partial x)*eq2-(\partial f_2/\partial y)*eq1)/wron,$$

where eq1 and eq2 are evaluated at our two points (-0.5,.25). After plugging in our x and y values, we should get (-.497,.254) as our new points(Hirst).

Because Newton's method involves linearization and solving repeatedly to find a solution to a nonlinear equation, we would continue calculating x and y until we are sufficiently close to the root (Hirst).