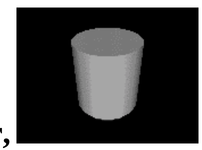
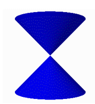
In Depth Mathematics

Since the two papers, <u>The Contact of a Cubic Surface with a Ruled Surface</u> and <u>Some Remarks on Ruled Surfaces</u>, published by Wilkins in the 1940's contain information about Ruled Surfaces and Cubic Surfaces, it is important that we understand these surfaces.

<u>Ruled Surfaces</u>

Cylinders and Cones

As was noted before, some basic examples of a ruled surface include cylinders and cones. These are classified as such because they are created by sweeping a straight line around a curve. A cylinder,  **for example, is formed when a normal line to a circle sweeps around the circumference of that circle. Cones are created by sweeping a line in a circular motion from a single point.**



It is simple to visualize the line that sweeps around the curve in both the cone and cylinder, but there are some other ruled surfaces that are not as easy to identify.

Hyperboloid and The Equation

There is an equation that can be used to discover whether or not a surface is ruled. The equation used to define a ruled surface is $x(s,v)=\alpha(t)+v*w(t)$, $t\in I$ and $v\in R$ where $\alpha(t)$ is the curve, and $w(t)$ is the vector which sweeps around the curve. It is easy to explore this equation by using an hyperboloid. The curve in the hyperboloid on which the line sweeps is a circle. In this case, we will use the unit circle, $x^2+y^2=1$. This can be written as $\alpha(s)=(\cos(s), \sin(s), 0)$. To find $w(s)$, we use the equation $w(s)= \alpha'(s) + e^3$ where e^3 is a unit vector of the z-axis. $\alpha'(s)=(-\sin(s), \cos(s), 0)$,

so $w(s)=(-\sin(s), \cos(s), 0) + (0,0,1) = (-\sin(s), \cos(s), 1)$. Going to the original equation, we get $x(s,t)=(\cos(s), \sin(s), 0)+v(-\sin(s), \cos(s), 1)$. By simplifying,

$x(s,t)=(\cos(s)-v*(\sin(s), \sin(s)+v*\cos(s),v)$. If we substitute these formulas for x,y, and z, in $x^2+y^2-z^2 = 1$, after, simplification, we get, $1+v^2-v^2$. Because $x^2+y^2-z^2=1$ and $1+v^2-v^2=1$, we know $x^2+y^2-z^2 = 1$ must be a ruled surface [2].

Here is a picture of a building that is an hyperboloid from Japan in the 1940's. 

Saddle Surface

Another ruled surface is a saddle surface, or hyperbolic paraboloid. Saddles surfaces are called this because their shape resembles that of a saddle used in riding horses or bicycles. The saddle equation, shown here, is defined as $kz=x^2+y\wedge^2$. A saddle is created by sweeping a line about a hyperbola. Another equation for a hyperbolic parablolid is z=kxy. The parametric equation of the saddle surface is created in much the same way as the hyperboloid. After taking the intersection of the family of curves in the z =0 plane, one gets $\alpha(t)=(t,0,0)$ and $w(t)=(0,1/k,t)$. By using the formula for a ruled surface, we get, $x(t,v)=(t,v/(\mathrm{sqrt}(1+k^2t^2)),vkt/(\mathrm{sqrt}(1+k^2t^2)))$. If we use the equation z=kxy to check the parametric equation, we get

$$vkt/(\mathrm{sqrt}(1+k^2t^2) = k* t*v/(\mathrm{sqrt}(1+k^2t^2)).$$

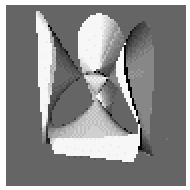
Since this is obviously a true statement, we know x(t,v) is an accurate parametric equation of this saddle surface. The following is a picture of a saddle surface.

Helicoid

The equation for the helicoid is y=x*tan(z/k). Visually, it is similar to the double helix form of DNA. The helicoid is created by sweeping a vector about a circle similar to the sweeping line of the hyperboloid. The difference is that at the same time as the line is rotating about the curve, the z values of the line are also increasing. In other words, the vector goes up as it sweeps around.

Cubic Surfaces

The other form of surfaces discussed in the papers by Wilkins is cubic surfaces. The cubic surface is a surface that can be defined by a third degree polynomial in three-dimensional space. One example of a cubic surface is x^3+y^3+z^3=1. Algebraic properties are used to study these figures. One important fact about cubic surfaces is the idea that each one has exactly twenty-seven straight lines on it. It is true, however that in regular, three-dimensional space, all of these lines cannot be easily seen. To "see" all of the lines on a surface, it is necessary to use a complex, three-dimensional space.



Papers

In his research, Wilkins used the definitions and theorems about both ruled and cubic surfaces to prove theorems about their relatedness. He used a general form of any ruled surface and at times in his second paper, dealt with Cayley's cubic. Wilkins explored the ways in which the two types of surfaces contacted each other. This included the number of places where they were in contact. Wilkins used power series expansion to prove the theorems in his papers.