Woodard's Mathematics

An In-Depth Look at the Jordan-curve Theorem

What does the Jordan-curve theorem state?

The idea behind the Jordan-curve theorem is that every simple closed curve divides the plane up so that there is one inside component and one outside component.

What exactly is a simple closed curve?

A proper explanation of this theorem must begin with the definition of a simple closed curve. A curve that is simple means that the line that defines its boundaries does not cross itself at any point along the curve. A curve that is closed begins and ends at the same point. So, the curve defined in the Jordan-curve theorem begins and ends at the same point and does not cross itself at any point along the curve.

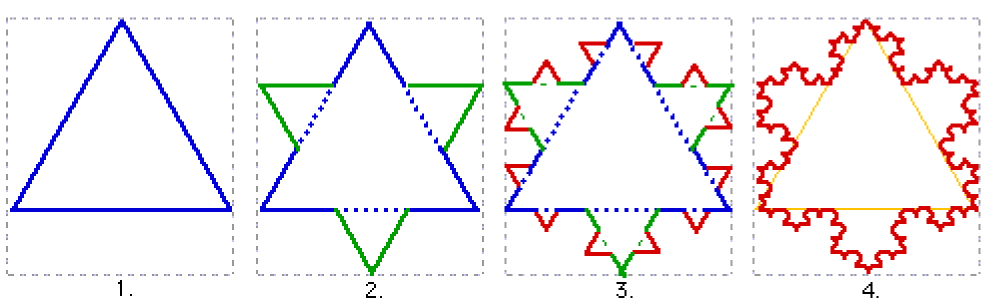
How did Woodard go about proving the Jordan-curve theorem?

R.L. Moore, another mathematician at the time, had developed some axioms that corresponded with the theorem. However, in these axioms Moore assumed some of the properties of a simple closed curve. In particular, Axiom 8 stated, “every simple closed curve is the boundary of a region, that is, every simple closed curve defines a bounded connected set of connected exterior having further properties implied by certain other axioms of the three systems."

Woodard's chief purpose for his investigation was to "replace Moore's Axiom 8 by another axiom of such nature that no property of the simple closed curve is assumed." He also omitted some of Moore's axioms that were not necessary to prove the others, and he also changed the wording in some of the others to make them clearer. After making these changes Woodard wrote, "It is of interest to note that under these circumstances, the proof of the Jordan-curve theorem is based upon a set of axioms which contain no assumption as to the character of the boundaries of regions (Woodard1, p.122)." Basically, Woodard's work on the Jordan curve theorem was geared toward eliminating all assumptions that Moore had made about the boundaries of these closed regions.

Why was it so hard to prove the Jordan-curve theorem? An Example

Here I will use the Koch Snowflake Curve, a well-known fractal, to demonstrate why the proof of this theorem would be so complicated. Following is a picture of the Koch Snowflake curve:



As you can see, the snowflake curve is drawn by starting with an equilateral triangle. On the middle third of each of the three sides, construct an equilateral triangle with sides that are 1/3 the length of the previous sides. Erase the base of each of the three new triangles. In the picture above, the dotted lines represent where the bases of the triangle that have been erased. Continue this process on each side of every new triangle. Notice that continuing this process infinitely many times forms the snowflake.

As you can see the snowflake formed has infinite perimeter and finite area. The infinite perimeter comes from adding infinitely many triangles to the outside edge of the curve. The aspect of finite area is a bit more interesting. If a circle is drawn around the original figure, the area of the figure remains inside the circle no matter how large the perimeter gets (rice). These are two interesting aspects of the Koch Snowflake.

Another interesting fact is that this image has no derivative at any given point. Most importantly, this snowflake is a simple closed curve because it does not cross itself, and you can pick a beginning and ending point that are the same point.

Embedded in the definition of a curve for the Jordan curve-theorem, is the criterion that you have a finite length curve. If we do we do the snowflake iteration 1000 times and compare it to the result of doing the iteration 1001 times, one would observe that it's really hard to tell when you are inside or outside of the curve. In fact, some points switch.

Based on this snowflake curve as our example, we can explain why the proof of the Jordan curve theorem is nontrivial. As the number of sides of the curve approaches infinity, it becomes more and more difficult to determine if any given point near the edge of the curve is located on the inside of the curve or on the outside of the curve.

Concluding Summary:

Through Woodard's attempts to solidify Miller's assumptions about the Jordan curve theorem, he was able to develop a concrete proof of the theorem. He was also one of the first to call this concept by the name Jordan curve-theorem. Although the Jordan curve theorem seems obvious, Woodard showed us that the proof was nontrivial. After all, his proof for this theorem was twenty-four pages long! This proof turned out to involve complex analysis and a lot of paper, work, and time!