# Fourier Series

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# The Gerneral Definition

Fourier tried expressing solutions in terms of trig functions to give analytic solutions to solve the heat equation.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)$$

# A bit on coefficients...

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx$$

# Complex Numbers

Another way to write the Fourier Series is as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{-2\pi inx}{L}} dx$$

$$e^{\frac{2\pi inx}{L}} = \cos\left(\frac{2\pi nx}{L}\right) + i \sin\left(\frac{2\pi nx}{L}\right)$$

$$c_n = \frac{1}{2}(a_n \pm ib_n)$$

# Example

Let's use the square wave:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ -1 & -\pi \leq x \leq 0 \end{cases}$$

Fourier Series:

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)x\right)}{2n+1}$$

Graphs Here!

# Gibbs Overshoot

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)x\right)}{2n+1} = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots\right)$$

Consider the partial sum...

$$\frac{\pi}{4}S_n(x) = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots + \frac{\sin((2n+1)x)}{2n+1}$$

$$\frac{\pi}{4}S_n'(x) = \cos(x) + \cos(3x) + \cos(5x) + \ldots \cos((2n+1)x)$$

Find the value of x where $S_n'(x) = 0$ near the origin. This would be $\frac{\pi}{2(n+1)}$ for the square wave example.

# How high is the Gibbs Overshoot?

$$S_n\left(\frac{\pi}{2n+2}\right) = \frac{4}{\pi}\sum_{k=0}^{n}\frac{\sin\left(\frac{\pi(2k+1)}{2n+2}\right)}{2k+1}$$

$$S_n \approx \frac{2}{\pi}\int_0^{\pi}\frac{\sin(t)}{t}dt \approx 1.1789797...$$

# Fourier Transform

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixk}dx$$

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$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x k} dx$$

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}(f) e^{-2\pi i x k} dk$$

# Convolution

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)dt$$

## Convolution Theorem

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{m} sinc\left(\frac{n}{m+1}\right)\left[a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)\right]$$

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