REVIEW EXERCISES FOR CHAPTER 1

Short answer exercises: Exercises 1–10 focus on the basic ideas, definitions, and vocabulary of this chapter. Their answers are short (a single sentence or drawing), and you should be able to do them with little or no computation. However, they vary in difficulty, so think carefully before you answer.

1. Give an example of a first-order differential equation that has the function $y(t) = 2t + 3$ as a solution.

2. What is the general solution of the differential equation $dy/dt = 3y$?

3. Find all equilibrium solutions for the differential equation $dy/dt = t^2(t^2 + 1)$.

4. Find one solution of the differential equation $dy/dt = -|\sin^5 y|$.

5. Find all of the equilibrium solutions for the differential equation

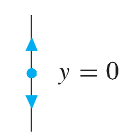
$$\frac{dy}{dt} = \frac{(t^2 - 4)(1 + y)e^y}{(t - 1)(3 - y)}.$$

6. Sketch the phase line for the autonomous equation $dy/dt = \sin^2 y$.

7. Give an example of a first-order differential equation that is autonomous, separable, linear, and homogeneous.

8. Give an example of a first-order, autonomous, linear, nonhomogeneous differential equation that has the equilibrium solution $y(t) = 2$ for all t.

9. Suppose the phase line to the right is the phase line for the autonomous differential equation $dy/dt = f(y)$. What can you say about the graph of $f(y)$?



10. What are the bifurcation values of the one-parameter family of differential equations $dy/dt = a + 4$?

True-false: For Exercises 11–20, determine if the statement is true or false. If it is true, explain why. If it is false, provide a counterexample or an explanation.

11. The function $y(t) = -e^{-t}$ is a solution to the differential equation $dy/dt = |y|$.

12. Every separable differential equation is autonomous.

13. Every autonomous differential equation is separable.

14. Every linear differential equation is separable.

15. Every separable differential equation is a homogeneous linear equation.

16. Every homogeneous linear differential equation is separable.

17. The solution of $dy/dt = (y - 3)(\sin y \sin t + \cos t + 1)$ with $y(0) = 4$ satisfies $y(t) > 3$ for all t.

18. Suppose that $f(y)$ is a continuous function for all y. The phase line for $dy/dt = f(y)$ must have the same number of sources as sinks.

19. Suppose that $f(y)$ is continuously differentiable for all y. Exactly one solution of $dy/dt = f(y)$ tends to ∞ as t increases.

20. Every solution of $dy/dt = y + e^{-t}$ tends to $+\infty$ or $-\infty$ as $t \to \infty$.

In Exercises 21–29,

(a) specify if the given equation is autonomous, linear and homogeneous, linear and nonhomogeneous, and/or separable, and

(b) find its general solution.

21. $\dfrac{dy}{dt} = 3 - 2y$

22. $\dfrac{dy}{dt} = ty$

23. $\dfrac{dy}{dt} = 3y + e^{7t}$

24. $\dfrac{dy}{dt} = \dfrac{ty}{1+t^2}$

25. $\dfrac{dy}{dt} = -5y + \sin 3t$

26. $\dfrac{dy}{dt} = t + \dfrac{2y}{1+t}$

27. $\dfrac{dy}{dt} = 3 + y^2$

28. $\dfrac{dy}{dt} = 2y - y^2$

29. $\dfrac{dy}{dt} = -3y + e^{-2t} + t^2$

In Exercises 30–39,

(a) specify if the given equation is autonomous, linear and homogeneous, linear and nonhomogeneous, and/or separable, and

(b) solve the initial-value problem.

30. $\dfrac{dx}{dt} = -2tx, \quad x(0) = e$

31. $\dfrac{dy}{dt} = 2y + \cos 4t, \quad y(0) = 1$

32. $\dfrac{dy}{dt} = 3y + 2e^{3t}, \quad y(0) = -1$

33. $\dfrac{dy}{dt} = t^2y^3 + y^3, \quad y(0) = -1/2$

34. $\dfrac{dy}{dt} + 5y = 3e^{-5t}, \quad y(0) = -2$

35. $\dfrac{dy}{dt} = 2ty + 3te^{t^2}, \quad y(0) = 1$

36. $\dfrac{dy}{dt} = \dfrac{(t+1)^2}{(y+1)^2}, \quad y(0) = 0$

37. $\dfrac{dy}{dt} = 2ty^2 + 3t^2y^2, \quad y(1) = -1$

38. $\dfrac{dy}{dt} = 1 - y^2, \quad y(0) = 1$

39. $\dfrac{dy}{dt} = \dfrac{t^2}{y + t^3y}, \quad y(0) = -2$

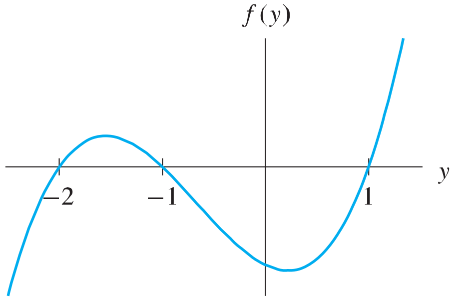
40. Consider the initial-value problem $dy/dt = y^2 - 2y + 1, \quad y(0) = 2$.

(a) Using Euler's method with $\Delta t = 0.5$, graph an approximate solution over the interval $0 \le t \le 2$.

(b) What happens when you try to repeat part (a) with $\Delta t = 0.05$?

(c) Solve this initial-value problem by separating variables, and use the result to explain your observations in parts (a) and (b).

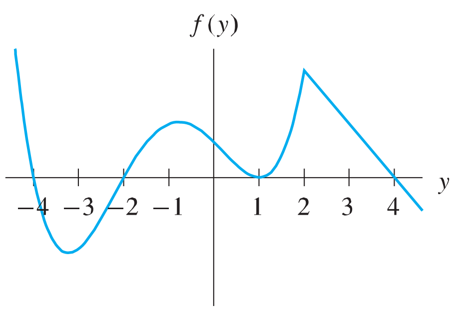
41. Consider the autonomous differential equation $dy/dt = f(y)$ where the graph of $f(y)$ is given below.



(a) Give a rough sketch of the slope field that corresponds to this equation.

(b) Give a rough sketch of the graph of the solution to $dy/dt = f(y)$ that satisfies the initial condition $y(0) = 0$.

42. Consider the autonomous differential equation $dy/dt = f(y)$ where the graph of $f(y)$ is given below.



(a) Sketch the phase line for this equation and identify the equilibrium points as sinks, sources, or nodes.

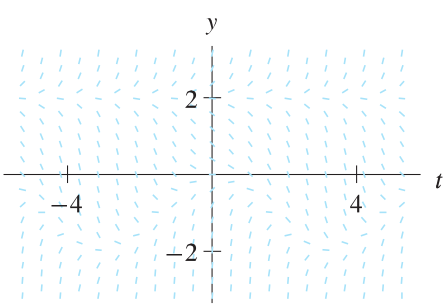
(b) Give a rough sketch of the slope field that corresponds to this equation.

(c) Give rough sketches of the graphs of the solutions that satisfy the initial conditions $y(0) = -3$, $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$.

43. The slope field to the right is the field for the differential equation

$$\frac{dy}{dt} = (y-2)(y+1-\cos t).$$

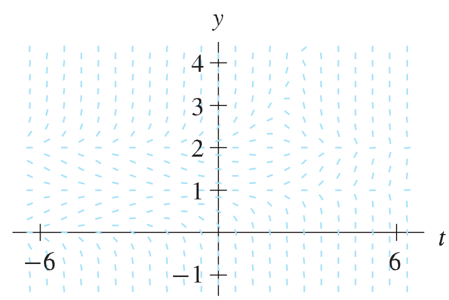
Describe the long-term behavior of solutions with various initial values at $t = 0$. Then confirm your answer with `HPGSolver`.



44. The slope field to the right is the field for the differential equation

$$\frac{dy}{dt} = (y-1)(y-2)(y-e^{t/2}).$$

Describe the long-term behavior of solutions with various initial values at $t = 0$. Then confirm your answer with HPGSolver.



45. Consider the differential equation

$$\frac{dy}{dt} = t^2y + 1 + y + t^2.$$

(a) Find its general solution by separating variables.

(b) Note that this equation is also a nonhomogeneous linear equation. Find the general solution of its associated homogeneous equation.

(c) Calculate the equilibrium solutions of the nonhomogeneous equation.

(d) Using the Extended Linearity Principle, find the general solution of the nonhomogeneous equation. Compare your result to the one you obtained in part (a).

46. Consider the differential equation

$$\frac{dy}{dt} = \frac{2y+1}{t}.$$

(a) Compute its general solution by separating variables.

(b) What happens to these solutions as $t \to 0$?

(c) Why doesn't this example violate the Uniqueness Theorem?

47. Consider the initial-value problem $dy/dt = 3 - y^2$, $y(0) = 0$.

(a) Using Euler's method with $\Delta t = 0.5$, plot the graph of an approximate solution over the interval $0 \le t \le 2$.

(b) Sketch the phase line for this differential equation.

(c) What does the phase line tell you about the approximate values that you computed in part (a)?

48. A cup of soup is initially 150°. Suppose that it cools to 140° in 1 minute in a room with an ambient temperature of 70°.

(a) Assume that Newton's law of cooling applies: The rate of cooling is proportional to the difference between the current temperature and the ambient temperature. Write an initial-value problem that models the temperature of the soup.

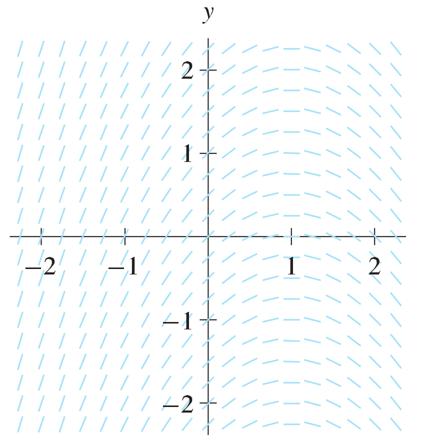
(b) How long does it take the soup to cool to a temperature of 100°?

49. Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field and state briefly how you know your choice is correct. You should do this exercise without using technology.

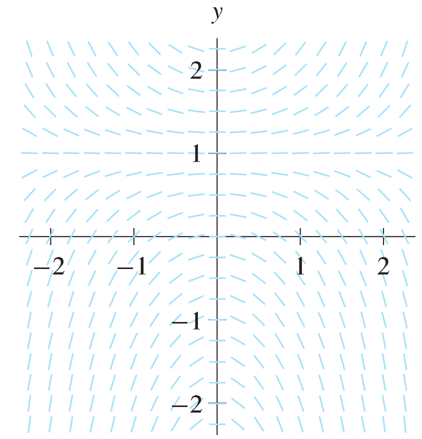
(i) $\dfrac{dy}{dt} = t - 1$ (ii) $\dfrac{dy}{dt} = 1 - y^2$ (iii) $\dfrac{dy}{dt} = y - t^2$ (iv) $\dfrac{dy}{dt} = 1 - t$

(v) $\dfrac{dy}{dt} = 1 - y$ (vi) $\dfrac{dy}{dt} = y + t^2$ (vii) $\dfrac{dy}{dt} = ty - t$ (viii) $\dfrac{dy}{dt} = y^2 - 1$

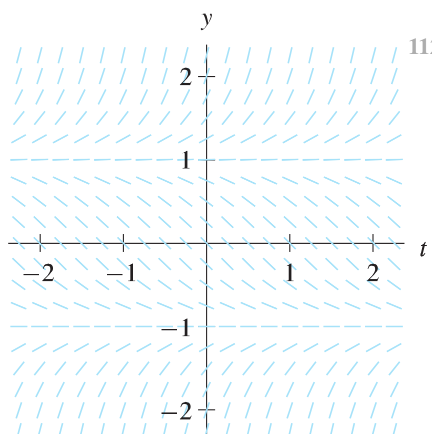
(a)



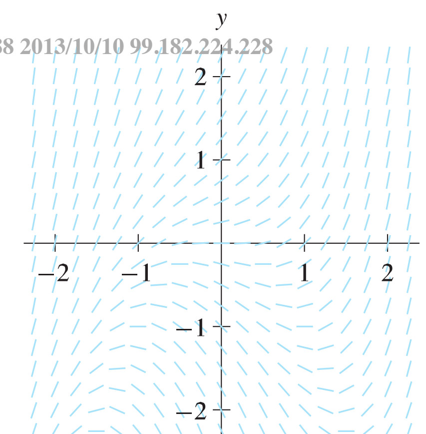
(b)



(c)



(d)



50. Beth initially deposits \$400 in a savings account that pays interest at the rate of 1.1% per year compounded continuously. She also arranges for \$20 per week to be deposited automatically into the account.

(a) Assume that weekly deposits are close enough to continuous deposits so that we can reasonably approximate her balance using a differential equation. Write an initial-value problem for her balance over time.

(b) Approximate Beth's balance after 4 years by solving the initial-value problem in part (a).

51. Consider the linear differential equation

$$a\frac{dy}{dt} + y = b,$$

where a and b are positive constants.

(a) Sketch the phase line associated with this equation.

(b) Describe the long-term behavior of all solutions.

(c) How many different methods do you know to calculate its general solution?

(d) Using your favorite method, calculate the general solution.

(e) Using your least favorite method, calculate the general solution.

(f) Using your answer in parts (d) and (e), confirm your answer to part (b).

52. Consider the differential equation $dy/dt = -2ty^2$.

(a) Calculate its general solution.

(b) Find all values of y_0 such that the solution to the initial-value problem

$$\frac{dy}{dt} = -2ty^2, \quad y(-1) = y_0,$$

does not blow up (or down) in finite time. In other words, find all y_0 such that the solution is defined for all real t.

53. The air in a small rectangular room 20 ft by 5 ft by 10 ft is 3% carbon monoxide. Starting at $t = 0$, air containing 1% carbon monoxide is blown into the room at the rate of 100 ft^3 per hour and well mixed air flows out through a vent at the same rate.

(a) Write an initial-value problem for the amount of carbon monoxide in the room over time.

(b) Sketch the phase line corresponding to the initial-value problem in part (a), and determine how much carbon monoxide will be in the room over the long term.

(c) When will the air in the room be 2% carbon monoxide?

54. A 1000-gallon tank initially contains a mixture of 450 gallons of cola and 50 gallons of cherry syrup. Cola is added at the rate of 8 gallons per minute, and cherry syrup is added at the rate of 2 gallons per minute. At the same time, a well mixed solution of cherry cola is withdrawn at the rate of 5 gallons per minute. What percentage of the mixture is cherry syrup when the tank is full?