REVIEW EXERCISES FOR CHAPTER 2

Short answer exercises: Exercises 1–14 focus on the basic ideas, definitions, and vocabulary of this chapter. Their answers are short (a single sentence or drawing), and you should be able to do them with little or no computation. However, they vary in difficulty, so think carefully before you answer.

1. Find one solution of the system $dx/dt = |x| \sin y$ and $dy/dt = |y| \cos x$.

2. Find all equilibrium points of the system $dx/dt = y$ and $dy/dt = e^y + x^2$.

3. Convert the second-order differential equation $d^2y/dt^2 = 1$ to a first-order system.

4. Find the general solution of the system of equations in Exercise 3.

5. Find all equilibrium points of the system $dx/dt = y$ and $dy/dt = \sin(xy)$.

6. How many equilibrium solutions does the system of differential equations $dx/dt = x(x - y)$ and $dy/dt = (x^2 - 4)(y^2 - 9)$ have? What are they?

7. Is the function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ a solution to the system of differential equations $dx/dt = 2x - 2y^2$ and $dy/dt = -3y$? Why?

8. Write the second-order equation and the corresponding first-order system for the mass-spring system with spring constant α, mass β, and damping coefficient γ.

9. Find the general solution of the system $dx/dt = 2x$ and $dy/dt = -3y$.

10. Sketch the $x(t)$- and $y(t)$-graphs corresponding to the solution of the initial-value problem $dx/dt = y^2 - 4$, $dy/dt = x^2 - 2x$, and $(x(0), y(0)) = (0, 2)$.

11. Give an example of a first-order system of differential equations with exactly ten equilibrium points.

12. Suppose that $\mathbf{F}(2, 1) = (3, 2)$. What is the result of one step of Euler's method applied to the initial-value problem $d\mathbf{Y}/dt = \mathbf{F}(\mathbf{Y})$, $\mathbf{Y}(0) = (2, 1)$, with $\Delta t = 0.5$?

13. Sketch the solution curve for the initial-value problem $dx/dt = -x$, $dy/dt = -y$, and $(x(0), y(0)) = (1, 1)$.

14. Suppose that all solutions of the system $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$ tend to an equilibrium point at the origin as t increases. What can you say about solutions of the system $dx/dt = -f(x, y)$ and $dy/dt = -g(x, y)$?

True-false: For Exercises 15–21, determine if the statement is true or false. If it is true, explain why. If it is false, provide a counterexample or an explanation.

15. The function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ is a solution to the system of differential equations $dx/dt = 2x - 2y^2$ and $dy/dt = -3y$.

16. The function $x(t) = 2$ for all t is an equilibrium solution of the system of differential equations $dx/dt = x - 2$ and $dy/dt = -y$.

17. Two different first-order autonomous systems can have the same vector field.

18. Two different first-order autonomous systems can have the same direction field.

19. The function $(x(t), y(t)) = (\sin t, \sin t)$ is a solution of some first-order autonomous system of differential equations.

20. If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution to an autonomous first-order system, then the function $(x_2(t), y_2(t)) = (\cos(t-1), \sin(t-1))$ is also a solution.

21. If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution of a first-order autonomous system, then the function $(x_2(t), y_2(t)) = (-\sin t, \cos t)$ is also a solution of the same system.

22. MacQuarie Island is a small island about half-way between Antarctica and New Zealand. As was mentioned in Exercise 11 of Section 1.1, its rabbit population underwent an explosion during the six-year period between 2000 and 2006. Before the year 2000, it was home to approximately 4,000 rabbits. It was also home to 160 feral cats and was an important nesting site for seabirds* The cats, being cats, attacked the nests of the seabirds. To protect the endangered birds, the cats were "eliminated" in 2000. However, the cats ate rabbits as well as seabirds. By 2006, the number of rabbits had grown to about 130,000.

Let $R(t)$ be the rabbit population and $C(t)$ be the cat population where time t is measured in years. Suppose the cat population is well approximated by a logistic model, while the rabbit population is modeled by a modified logistic model. We use

$$\frac{dC}{dt} = C\left(1 - \frac{C}{160}\right)$$

$$\frac{dR}{dt} = R\left(1 - \frac{R}{130{,}000}\right) - \alpha RC,$$

where the $-\alpha RC$ term measures the negative effects on the rabbits during their interactions with the cats.

(a) What value of α gives an equilibrium point at $C = 160$ and $R = 4000$?

(b) Using the value of α from part (a), calculate the contribution that the term $-\alpha RC$ makes to dR/dt when $C = 160$ and $R = 4000$. Assuming that this value represents the decrease in the rabbit population per year caused by the cats, approximately how many rabbits did each cat eliminate per year (when $C = 160$ and $R = 4000$)?

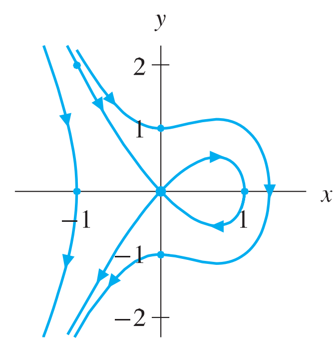
(c) A plan is being developed to "remove" the rabbits and other rodents. Could the rabbit population be controlled by instituting a constant harvesting parameter? If so, how many rabbits would have to be harvested per year?

*See "Rampant rabbits trash World Heritage island" by Rachel Nowak *New Scientist*, January 14, 2009. Available at www.newscientist.com

True-false: Solution curves for several solutions of the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = 2x - 3x^2$$

are shown in the figure below. For Exercises 23–28, determine if the statement is true or false for this system. Justify your answer.



23. The solution curve corresponding to the initial condition $(1, 0)$ includes the point $(0, 0)$.

24. The $x(t)$- and $y(t)$-graphs of the solution with $(x(0), y(0)) = (1/2, 0)$ tend to infinity as t increases.

25. The solution with initial condition $(x(0), y(0)) = (0, 1)$ is the same as the solution with initial condition $(x(0), y(0)) = (0, -1)$.

26. The function $y(t)$ for the solution with initial condition $(x(0), y(0)) = (-1, 2)$ is positive for all $t > 0$.

27. The functions $x(t)$ and $y(t)$ for the solution with initial condition $(x(0), y(0)) = (-1, 0)$ decrease monotonically for all t.

28. The $x(t)$- and $y(t)$-graphs of the solution with initial condition $(x(0), y(0)) = (0, 1)$ each have exactly one critical point.

29. Consider the system

$$\frac{dx}{dt} = \cos 2y$$
$$\frac{dy}{dt} = 2y - x.$$

(a) Find its equilibrium points.

(b) Use `HPGSystemSolver` to plot its direction field and phase portrait.

(c) Briefly describe the behavior of typical solutions.

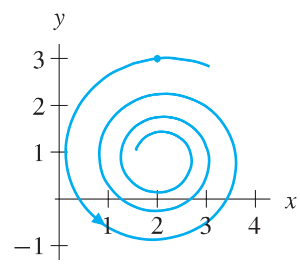
30. Consider a decoupled system of the form

$$\frac{dx}{dt} = f(x)$$
$$\frac{dy}{dt} = g(y).$$

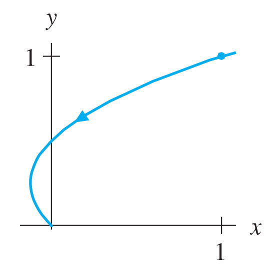
What special features does the phase portrait of this system have?

In Exercises 31–34, a solution curve in the xy-plane and an initial condition on that curve are specified. Sketch the $x(t)$- and $y(t)$-graphs for the solution.

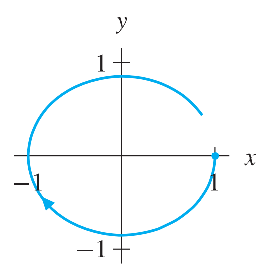
31.



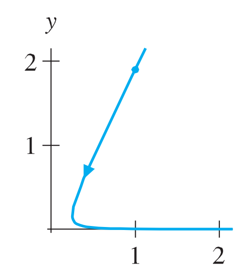
32.



33.



34.



35. Consider the partially decoupled system

$$\frac{dx}{dt} = x + 2y + 1$$
$$\frac{dy}{dt} = 3y.$$

(a) Derive the general solution.

(b) Find the equilibrium points of the system.

(c) Find the solution that satisfies the initial condition $(x_0, y_0) = (-1, 3)$.

(d) Use `HPGSystemSolver` to plot the phase portrait for this system. Identify the solution curve that corresponds to the solution with initial condition $(x_0, y_0) = (-1, 3)$.

36. Consider the partially decoupled system

$$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = y + 1.$$

(a) Derive the general solution.

(b) Find the equilibrium points of the system.

(c) Find the solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$.

(d) Use `HPGSystemSolver` to plot the phase portrait for this system. Identify the solution curve that corresponds to the solution with initial condition $(x_0, y_0) = (1, 0)$.

37. A simple model of a glider flying along up and down but not left or right ("planar" motion) is given by

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s}$$

$$\frac{ds}{dt} = -\sin\theta - Ds^2,$$

where θ represents the angle of the nose of the glider with the horizon, $s > 0$ represents its speed, and $D \geq 0$ is a parameter that represents drag (see the `DETools` program `HMSGlider`).

(a) Calculate the equilibrium points for this system.

(b) Give a physical description of the motion of the glider that corresponds to these points.