REVIEW EXERCISES FOR CHAPTER 3

Short answer exercises: Exercises 1–10 focus on the basic ideas, definitions, and vocabulary of this chapter. Their answers are short (a single sentence or drawing), and you should be able to do them with little or no computation. However, they vary in difficulty, so think carefully before you answer.

1. What are the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}?$$

2. What are the eigenvalues of the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}?$$

3. Compute the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{Y}$$

and sketch its phase portrait.

4. Which of the following vectors are eigenvectors for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}?$$

(i) $\mathbf{Y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (ii) $\mathbf{Y}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ (iii) $\mathbf{Y}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(iv) $\mathbf{Y}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (v) $\mathbf{Y}_5 = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ (vi) $\mathbf{Y}_6 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

5. Consider the harmonic oscillator with mass 1, spring constant 5, and damping coefficient b. Find the values of b for which the system is overdamped, underdamped, critically damped, or undamped.

6. Compute the equilibrium points for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

7. Solve the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} \pi^2 & 37.4 \\ \sqrt{555} & 8.01234 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

8. For each function $y(t)$, determine if it is a solution to the second-order equation

$$\frac{d^2y}{dt^2} + ky = 0,$$

assuming k satisfies the given condition.

(a) $y(t) = \sin kt,\ k < 0$
(b) $y(t) = 0,\ k > 0$
(c) $y(t) = t^2,\ k = 0$
(d) $y(t) = \sin \sqrt{k}\,t + 2\cos \sqrt{k}\,t,\ k > 0$
(e) $y(t) = e^{kt},\ k > 0$
(f) $y(t) = e^{\sqrt{-kt}},\ k < 0$

9. Find a linear system for which the function $\mathbf{Y}(t) = (3\cos 2t, \sin 2t)$ is a solution.

10. Compute the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$$

and sketch its phase portrait.

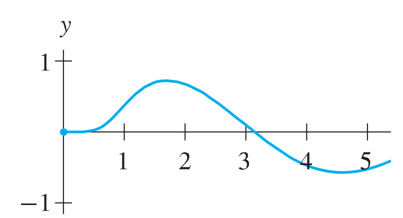
True-false: For Exercises 11–18, determine if the statement is true or false. If it is true, explain why. If it is false, provide a counterexample or an explanation.

11. The origin is the only equilibrium point for any linear system.

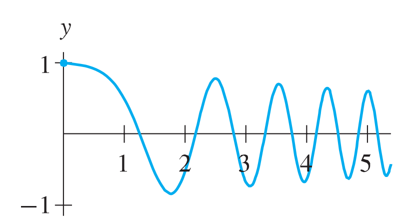
12. If $\mathbf{Y}_0$ is an eigenvector for a matrix, then so is any nonzero scalar multiple of $\mathbf{Y}_0$.

13. The function $\mathbf{Y}(t) = (\cos 2t, \sin t)$ is not a solution to any linear system.

14. The graph on the right is the graph of the solution of a damped harmonic oscillator.



15. The graph on the right is the graph of a typical solution of a damped harmonic oscillator.



16. If k increases, then the time between successive maxima of solutions of the harmonic oscillator $d^2y/dt^2 + ky = 0$ decreases.

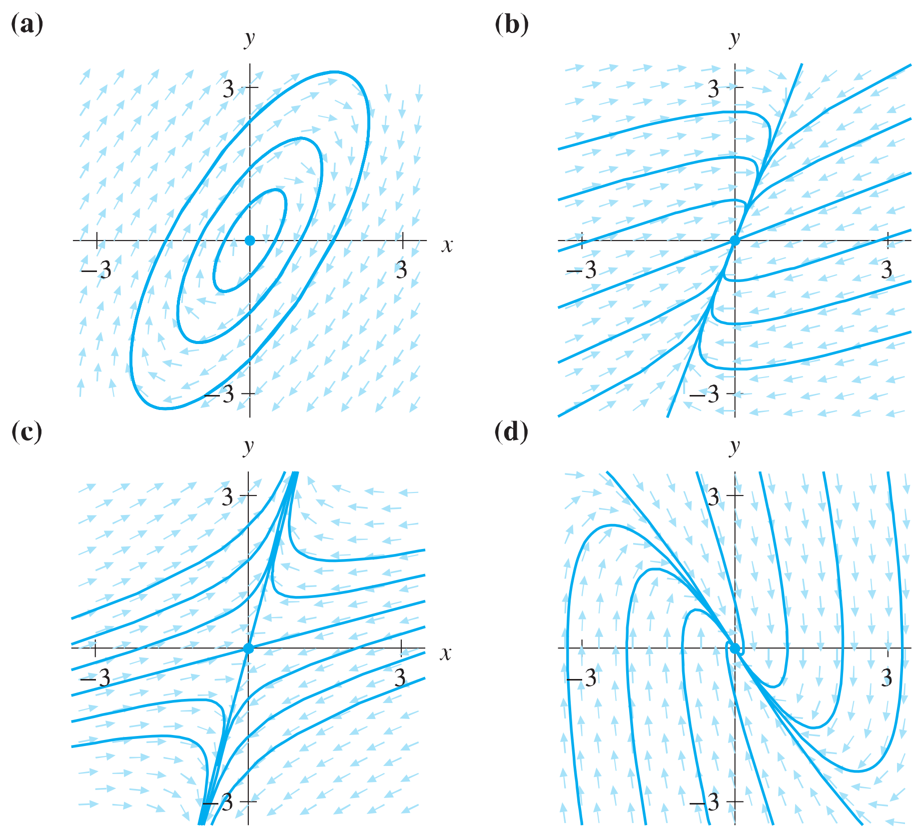
17. Suppose $\mathbf{A}$ is a 2×2 matrix. The linear system $d\mathbf{Y}/dt = \mathbf{AY}$ can have three different straight-line solutions.

18. Suppose $\mathbf{A}$ is a 2×2 matrix. No solution of $d\mathbf{Y}/dt = \mathbf{AY}$ can blow up in finite time.

19. Eight matrices and four phase portraits are given below. For each matrix, form the associated linear system, and determine which system corresponds to each phase portrait. State briefly how you know your choice is correct. You should do this exercise without using technology.

(i) $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ (vi) $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ (vii) $\begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$ (viii) $\begin{pmatrix} -3 & -3 \\ 2 & 1 \end{pmatrix}$



20. Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3a \\ 1 & a \end{pmatrix} \mathbf{Y}.$$

(a) Draw the curve in the trace-determinant plane that is obtained from varying the parameter a.

(b) Determine all bifurcation values of a and briefly discuss the different types of phase portraits that are exhibited in this one-parameter family.

21. Eight matrices and four pairs of $x(t)$- and $y(t)$-graphs of solutions to linear systems are given below. For each matrix, form the associated linear system, and determine which system corresponds to each pair of graphs. State briefly how you know your choice is correct. You should do this exercise without using technology.

(i) $\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$

(iii) $\begin{pmatrix} -3 & -2 \\ 0 & -1 \end{pmatrix}$

(iv) $\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$

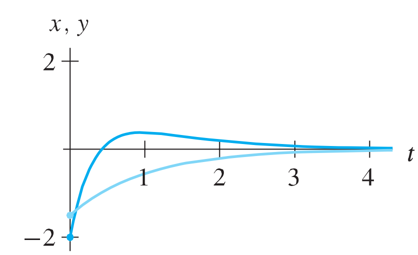
(v) $\begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$

(vi) $\begin{pmatrix} 1 & .25 \\ .25 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} -1.1 & -2 \\ 2 & -1.1 \end{pmatrix}$

(viii) $\begin{pmatrix} -1.1 & -5 \\ 1 & 0.9 \end{pmatrix}$

(a)



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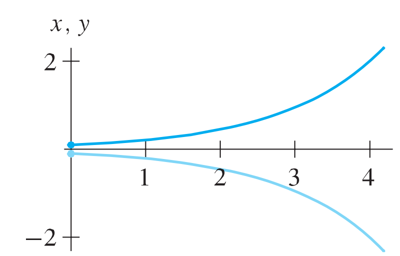

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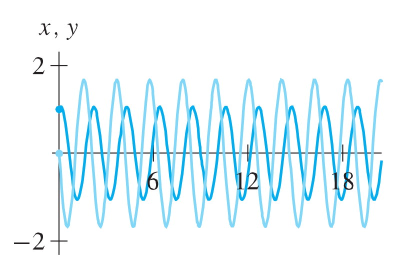

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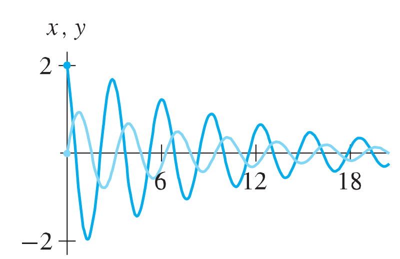

22. **(a)** Give an example of a linear system that has an equilibrium solution $(x(t), y(t))$ such that $x(0) = -1$ and $y(t) = 2x(t)$ for all t.

(b) Give an example of a linear system that has a straight-line solution $(x(t), y(t))$ such that $x(0) = -1$ and $y(t) = 2x(t)$ for all t.

In Exercises 23–26, find the solution (in scalar form) of the given initial-value problem.

23. $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$
$y(0) = 0,\ y'(0) = 2$

24. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$
$y(0) = 3,\ y'(0) = -1$

25. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$
$y(0) = 1,\ y'(0) = 1$

26. $\frac{d^2y}{dt^2} + 2y = 0$
$y(0) = 3,\ y'(0) = -\sqrt{2}$

In Exercises 27–32, a linear system and an initial condition are given. For each system,

(a) compute the general solution;
(b) sketch its phase portrait;
(c) solve the initial-value problem; and
(d) sketch the $x(t)$- and $y(t)$-graphs for the solution to the initial-value problem.

27. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

28. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

29. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ -2 & 2 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

30. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$

31. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

32. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$