REVIEW EXERCISES FOR CHAPTER 5

Short answer exercises: Exercises 1–10 focus on the basic ideas, definitions, and vocabulary of this chapter. Their answers are short (a single sentence or drawing), and you should be able to do them with little or no computation. However, they vary in difficulty, so think carefully before you answer.

1. For the nonlinear system $dx/dt = x + xy$ and $dy/dt = y^2 - 2y$, what is the linearized system for the equilibrium point at the origin?

2. For the nonlinear system in Exercise 1, determine the type (sink, source, ...) of the equilibrium point at the origin.

3. For the nonlinear system $dx/dt = x^2 + \sin 3x$ and $dy/dt = 2y - \sin xy$, what is the linearized system for the equilibrium point at the origin?

4. For the nonlinear system in Exercise 3, determine the type (sink, source, ...) of the equilibrium point at the origin.

5. Sketch the nullclines for the system $dx/dt = x - y$ and $dy/dt = x^2 + y^2 - 2$.

6. Is the system $dx/dt = 2xy + y^2$ and $dy/dt = x^2 + y^2$ a Hamiltonian system?

7. Is the system in Exercise 6 a gradient system?

8. Describe three different possible long-term behaviors for a solution $\mathbf{Y}(t)$ of a two-dimensional system $d\mathbf{Y}/dt = \mathbf{F}(\mathbf{Y})$.

9. Suppose that a two-dimensional system $d\mathbf{Y}/dt = \mathbf{F}(\mathbf{Y})$ has exactly one equilibrium point and that point is a source. Describe two different possible long-term behaviors for a solution with an initial condition that is near the equilibrium point.

10. Suppose that the origin is an equilibrium point of a two-dimensional system and that zero is not an eigenvalue for the linearized system at the origin. Suppose also that $dy/dt > 0$ on the positive x-axis and $dx/dt > 0$ on the positive y-axis. What type of equilibrium point can the origin be?

True-false: For Exercises 11–14, determine if the statement is true or false. If it is true, explain why. If it is false, provide a counterexample or an explanation.

11. The equilibrium points of a first-order system occur at the intersection of the x- and y-nullclines.

12. The x- and y-nullclines of a system are never identical.

13. If (x_0, y_0) is an equilibrium point of the system $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$, then the eigenvalues of the linearized system at (x_0, y_0) are the partial derivatives $\partial f/\partial x$ and $\partial g/\partial y$ evaluated at (x_0, y_0).

14. For a system of the form $dx/dt = f(x)$ and $dy/dt = g(x, y)$ with $f'(x) > 0$ for all x, every equilibrium point is a source.

For the systems given in Exercises 15–18,

(a) find and classify all equilibria,

(b) sketch the nullclines, and

(c) sketch the phase portrait.

15. $$\frac{dx}{dt} = x - 3y^2$$
$$\frac{dy}{dt} = x - 3y - 6$$

16. $$\frac{dx}{dt} = 10 - x^2 - y^2$$
$$\frac{dy}{dt} = 3x - y$$

17. $$\frac{dx}{dt} = 4x - x^2 - xy$$
$$\frac{dy}{dt} = 6y - 2y^2 - xy$$

18. $$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = x + y - 1$$

Exercises 19–24 concern the two-parameter family of systems

$$\frac{dx}{dt} = x^2 - a$$
$$\frac{dy}{dt} = y^2 - b,$$

where a and b are parameters.

19. Find and classify all equilibrium points assuming that $a > 0$ and $b > 0$.

20. Sketch the nullclines and the phase portrait assuming that $a > 0$ and $b > 0$.

21. Repeat Exercises 19 and 20 assuming that $a = 0$ and $b > 0$. In addition, describe the bifurcation that occurs as $a \to 0$ from above.

22. Repeat Exercises 19 and 20 assuming that $a > 0$ and $b = 0$. In addition, describe the bifurcation that occurs as $b \to 0$ from above.

23. Repeat Exercises 19 and 20 assuming that $a = 0$ and $b = 0$.

24. Draw a picture in the ab-plane that represents the different types of phase portraits that occur for this system for all values of a and b.

25. Consider the system

$$\frac{dx}{dt} = y^2 - x^2 - 1$$
$$\frac{dy}{dt} = 2xy.$$

(a) Find and classify all equilibrium points.

(b) Sketch the nullclines and phase portrait.

(c) Is this system a Hamiltonian system? If so, calculate the Hamiltonian function.

(d) Is this system a gradient system? If so, calculate a function whose gradient yields the system.

26. Consider the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0.$$

(a) Convert this second-order equation into a first-order system and calculate its equilibrium points.

(b) Classify these equilibrium points using linearization.

27. In Exercise 26 of Section 3.4, we showed that the solution curves of the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 10 \\ -1 & 3 \end{pmatrix} \mathbf{Y}$$

lie on ellipses of the form $x^2 - 6xy + 10y^2 = k$, where k is a constant determined by the initial condition. Reproduce that result by calculating a Hamiltonian function for the system.

28. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y}.$$

(a) For which values of a, b, c, and d is the system Hamiltonian? Calculate the Hamiltonian function for these values.

(b) For which values of a, b, c, and d is the system a gradient system? For these values, calculate a function whose gradient yields the system.

(c) Can the linear system be Hamiltonian and a gradient system for identical values of a, b, c, and d?

(d) Are there values of a, b, c, and d for which the linear system is neither Hamiltonian nor a gradient system?

29. In Exercise 37 of the Review Exercises of Chapter 2, we studied a simple model of a glider. Setting the drag parameter $D = 1$, we have the equations

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s}$$

$$\frac{ds}{dt} = -\sin\theta - s^2,$$

where θ represents the angle of the nose of the glider with the horizon and $s > 0$ represents its speed.

(a) Calculate the equilibrium points for this system, and

(b) classify them using linearization.