REVIEW EXERCISES FOR CHAPTER 6

Short answer exercises: Exercises 1–10 focus on the basic ideas, definitions, and vocabulary of this chapter. Their answers are short (a single sentence or drawing), and you should be able to do them with little or no computation. However, they vary in difficulty, so think carefully before you answer.

1. Calculate $\mathcal{L}^{-1}\left[\dfrac{4}{s^2-1}\right]$.

2. Find a formula for $\mathcal{L}\left[\dfrac{d^3y}{dt^3}\right]$ in terms of $\mathcal{L}[y]$.

3. What is $\mathcal{L}[y]$ for the solution $y(t)$ to the initial-value problem

$$\frac{d^2y}{dt^2} + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0?$$

4. Calculate $\displaystyle\int_0^\infty (1 - u_4(t))t\,dt$.

5. Calculate $\mathcal{L}[y]$ for the function

$$y(t) = \begin{cases} 1, & \text{if } t < 6; \\ 0, & \text{if } t \geq 6. \end{cases}$$

6. Write the function

$$y(t) = \begin{cases} 1, & \text{if } t < 1; \\ \sin t, & \text{if } 1 \leq t < \pi; \\ 2, & \text{if } t \geq \pi \end{cases}$$

in terms of the Heaviside function.

7. What is the value of the improper integral $\displaystyle\int_0^\infty \delta_2(t)e^{-2t}\,dt$?

8. Solve the initial-value problem $dy/dt = \delta_1(t), \quad y(0) = 0$.

9. Solve the initial-value problem

$$\frac{dy}{dt} = \delta_1(t) + \delta_2(t) + \delta_3(t) + \dots, \quad y(0) = 0,$$

and sketch the graph of $y(t)$.

10. Solve the initial-value problem

$$\frac{dy}{dt} = \delta_1(t) - \delta_2(t) + \delta_3(t) - \delta_4(t) + \dots, \quad y(0) = 0,$$

and sketch the graph of $y(t)$.

11. Six functions $y(t)$ and twelve functions $Y(s)$ are given below. For each function $y(t)$, match it with its Laplace transform $Y(s)$. You should do this exercise without consulting the table of Laplace transforms.

(a) $y(t) = \sin 2t$ **(b)** $y(t) = e^{2t}\cos 2t$ **(c)** $y(t) = e^{2t} - e^{-2t}$

(d) $y(t) = \sin(t - \pi)$ **(e)** $y(t) = \cos 2t$ **(f)** $y(t) = u_2(t)e^{3(t-2)}$

(i) $Y(s) = \dfrac{4}{s^2 - 4}$ (ii) $Y(s) = \dfrac{4}{s^2 + 4}$ (iii) $Y(s) = \dfrac{2}{s^2 + 4}$

(iv) $Y(s) = \dfrac{2s}{s^2 + 4}$ (v) $Y(s) = \dfrac{s}{s^2 - 4}$ (vi) $Y(s) = \dfrac{-1}{s^2 + 1}$

(vii) $Y(s) = \dfrac{s}{s^2 + 4}$ (viii) $Y(s) = \dfrac{e^{-2s}}{s + 3}$ (ix) $Y(s) = \dfrac{e^{-2s}}{s - 3}$

(x) $Y(s) = \dfrac{s}{s^2 - 4s + 8}$ (xi) $Y(s) = \dfrac{s - 2}{s^2 - 4s + 8}$ (xii) $Y(s) = \dfrac{e^{-\pi s}}{s^2 + 1}$

True-false: For Exercises 12–17, determine if the statement is true or false. If it is true, explain why. If it is false, provide a counterexample or an explanation.

12. The Laplace transform turns differentiation into multiplication.

13. The Laplace transform of the function $\delta_1(t)$ is a differentiable function.

14. If $y_1(t)$ is the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 6y = \delta_3(t), \quad y(0) = 0, \quad y'(0) = 0,$$

then $2y_1(t)$ is the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 6y = \delta_6(t), \quad y(0) = 0, \quad y'(0) = 0.$$

15. The solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 4y = \delta_\pi(t) + \delta_{2\pi}(t) + \delta_{4\pi}(t) + \dots, \quad y(0) = 0, \quad y'(0) = 0$$

oscillates with increasing and unbounded amplitude.

16. The function

$$y(t) = \mathcal{L}^{-1}\left[\frac{7s}{(s^2 + 1)(s^2 + 3)(s^2 + 5)}\right]$$

satisfies $y(t) \to \infty$ as $t \to \infty$.

17. The function

$$y(t) = \mathcal{L}^{-1}\left[\frac{3s+5}{s^4+6s^2+9}\right]$$

satisfies $-15 \le y(t) \le 15$ for all t.

In Exercises 18–23, find the inverse Laplace transform of the given function.

18. $\dfrac{3}{s^2+5s+6}$

19. $\dfrac{s+16}{s^2+2s-8}$

20. $\dfrac{2s+3}{s^2-2s+4}$

21. $\dfrac{(5s-12)e^{-3s}}{s^2-5s+6}$

22. $\dfrac{5s^2-27s+49}{(s-2)(s^2-6s+13)}$

23. $\dfrac{1}{s^2-4s+4}$

Given a forced, second-order, linear equation such as we studied in Chapter 4, we now have two ways to solve it. We can use the Extended Linearity Principle along with a guessing technique or we can use the Laplace transform. For the initial-value problems given in Exercises 24 and 25,

(a) find the solution using the method of Chapter 4,

(b) find the solution using the Laplace transform, and

(c) explain which method you prefer.

24. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} = 4, \quad y(0) = -1, \quad y'(0) = 2$

25. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 5$

In Exercises 26–30, solve the given initial-value problem.

26. $\dfrac{dy}{dt} - 3y = h(t), \quad y(0) = 0, \quad \text{where} \quad h(t) = \begin{cases} 6, & \text{if } t < 3; \\ 0, & \text{if } t \ge 3. \end{cases}$

27. $\dfrac{dy}{dt} - 4y = 50u_2(t)\sin(3(t-2)), \quad y(0) = 5$

28. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 7y = h(t), \quad y(0) = 3, \quad y'(0) = 0, \quad \text{where} \quad h(t) = \begin{cases} 1, & \text{if } t < 3; \\ 0, & \text{if } t \ge 3. \end{cases}$

29. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = \delta_4(t), \quad y(0) = 2, \quad y'(0) = 1$

30. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = \delta_3(t) + u_6(t), \quad y(0) = 1, \quad y'(0) = 2$