A Summary of Linear, Constant-Coefficient, Homogenous, Second Order Systems of Differential Equations

Sections

Two Real Eigenvalues

Real Distinct Nonzero Eigenvalues $\lambda_1 \neq \lambda_2$

For the system $\frac{d}{dt}\vec{Y} = \mathbf{A}\vec{Y}$:

1. Determine $\mathbf{A}$'s eigenvalues, λ_1 and λ_2, the roots of the *characteristic polynomial* $\mathrm{Char}(\mathbf{A}) = \det(\mathbf{A} - \lambda\mathbf{I})$.

2. Determine an *eigenvector* $\vec{v}_i$ for each eigenvalue λ_i by solving $(\mathbf{A} - \lambda_i\mathbf{I})\vec{v} = \vec{0}$ for $\vec{v}$ choosing convenient values for the parameter.

3. Then the general solution to the differential equation is

$$\vec{Y}(t) = c_1 e^{\lambda_1 t}\,\vec{v}_1 + c_2 e^{\lambda_2 t}\,\vec{v}_2$$

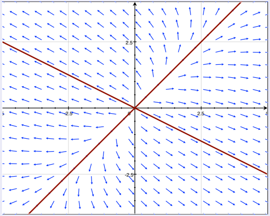
4. The two *"straight line"* solutions to the differential equation are

$$\vec{L}_1 = \vec{v}_1\,t \qquad \text{and} \qquad \vec{L}_2 = \vec{v}_2\,t \qquad \text{for } t \in \mathbb{R}$$

λ's Possibilities

I. $0 < \lambda_1 < \lambda_2$

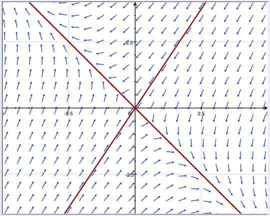
$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$



Source, Unstable

II. $\lambda_1 < 0 < \lambda_2$

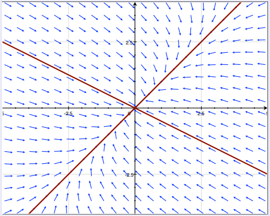
$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ -3 & -2 \end{bmatrix}$$



Saddle, Unstable

III. $\lambda_1 < \lambda_2 < 0$

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$$



Sink, Stable

A Repeated Eigenvalue $\lambda \neq 0$

$\lambda \neq 0$

λ is a root of multiplicity 2 of $\mathbf{A}$'s characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Case I. $\mathbf{A} = \lambda\mathbf{I}$.

- Every vector is an eigenvector
- The general solution is:

$$\vec{Y}(t) = c_1 e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{\lambda t}\begin{bmatrix}0\\1\end{bmatrix} = e^{\lambda t}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

- Star node: pos/neg λ gives unstable/stable

Case II. $\mathbf{A} \neq \lambda\mathbf{I}$.

- One eigenvector $\vec{v}_1$, so only one straight-line solution.
- The general solution requires a "generalized eigenvector" $\vec{v}_2$: solve $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{v}_1$. Then

$$\vec{Y}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 t\, e^{\lambda t}\vec{v}_2$$

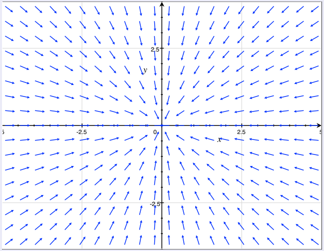
- Hyperbolic Star node: pos/neg λ gives unstable/stable

λ's Possibilities: Star Node

Case I. $\mathbf{A} = \lambda \mathbf{I}$

Stable: $\lambda < 0$

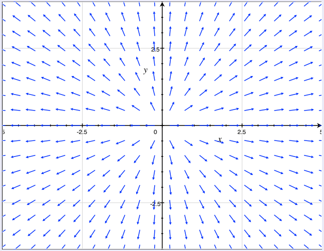
$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$



Sink, Stable

Unstable: $\lambda > 0$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



Source, Unstable

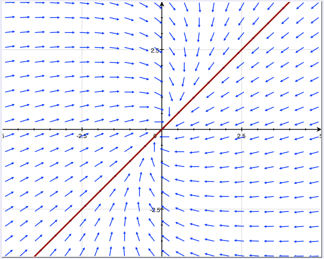
Every line through $\vec{0}$ is a straight line solution.

λ's Possibilities: Hyperbolic Star Node

Case II. $\mathbf{A} \neq \lambda \mathbf{I}$

Stable: $\lambda < 0$

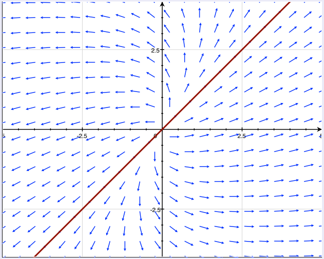
$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$



Sink, Stable

Unstable: $\lambda > 0$

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$



Source, Unstable

A Repeated Zero Eigenvalue

$\lambda_1 = 0$ and $\lambda_2 = 0$

$\mathbf{A} = \mathbf{0}$: The trivial system solution is $\vec{Y}(t) = \vec{c}$. Every point is an equilibrium.

$\mathbf{A} \neq \mathbf{0}$: Then $\det(\mathbf{A}) = 0$ and $\mathrm{Tr}(\mathbf{A}) = 0$ gives (if $a_{1,1} \neq 0$)[1]

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ -a_{1,1}^2/a_{1,2} & -a_{1,1} \end{bmatrix}$$

$\mathbf{A}$'s single eigenvector is $\vec{v} = \begin{bmatrix} -a_{1,2} \\ a_{1,1} \end{bmatrix}$
The general solution is

$$\vec{Y}(t) = (c_1 + c_2 t)\vec{v} + \vec{v}_0$$

where $\vec{v}_0$ is a constant vector. $\vec{Y}(t)$ reduces to a line.

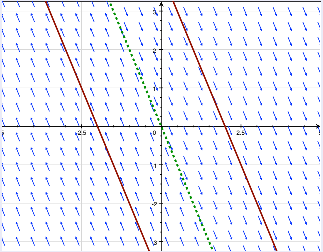
[1] If $a_{1,1} = 0$, then one of $a_{1,2}$ and $a_{2,1}$ must be nonzero and the other 0.

λ's Possibilities: Double Zero

$\mathbf{A} \neq \mathbf{0}$ but $\lambda_1 = 0, \lambda_2 = 0$

Unstable: $\lambda_1 = 0 = \lambda_2$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$



Line ($y = -2x$) of unstable equilibrium points.

Every nontrivial solution is a straight line parallel to the line of equilibria.

A Zero and a Nonzero Eigenvalue $\lambda \neq 0$

$\lambda_1 = 0$ and $\lambda_2 \neq 0$

If $\mathbf{A} \neq \mathbf{0}$ and $\det(\mathbf{A}) = 0$:

- $\lambda_1 = 0$ and $\lambda_2 = \mathrm{Tr}(\mathbf{A}) \neq 0$.
- An eigenvector for $\lambda_1 = 0$ is $\vec{v}_1 = \begin{bmatrix} -a_{1,2} \\ a_{1,1} \end{bmatrix}$.
- Then $\vec{L} = \begin{bmatrix} -a_{1,2} \\ a_{1,1} \end{bmatrix} t$ is a line of equilibrium points.
- An eigenvector for $\lambda_2 = \mathrm{Tr}(\mathbf{A})$ is $\vec{v}_2 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$
- The general solution is

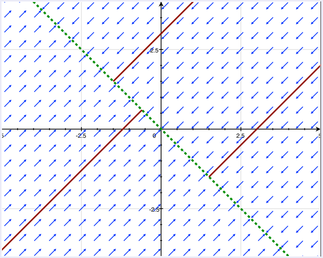
$$\vec{Y}(t) = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 \begin{bmatrix} -a_{1,2} \\ a_{1,1} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$$

λ's Possibilities: A Zero and a Nonzero λ

$\lambda_1 = 0$ and $\lambda_2 \neq 0$

Stable: $\lambda_2 < 0$

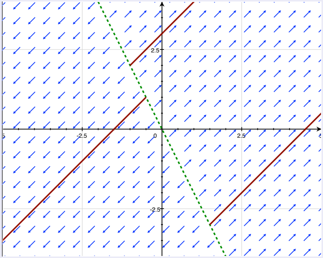
$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$



Line ($y = -x$) of Sinks

Unstable: $\lambda_2 > 0$

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$



Line ($y = -2x$) of Sources

Complex Conjugate Eigenvalues λ and $\bar{\lambda}$

$\lambda = a + b\imath$ and $\bar{\lambda} = a - b\imath$

- The eigenvalues are complex when $\mathrm{Tr}(\mathbf{A})^2 < 4\det(\mathbf{A})$

$$\lambda_{\pm} = \tfrac{1}{2}\,\mathrm{Tr}(\mathbf{A}) \pm \tfrac{1}{2}\,\imath\sqrt{4\det(\mathbf{A}) - \mathrm{Tr}(\mathbf{A})^2}$$

- When $\lambda_{\pm} = a \pm b\imath$, Euler's formula[2] implies the general solution is

$$\begin{aligned}\vec{Y}(t) &= \vec{c}_1 e^{at}\cos(bt) + \vec{c}_2 e^{at}\sin(bt) \\ &= e^{at}\left(\vec{c}_1\cos(bt) + \vec{c}_2\sin(bt)\right)\end{aligned}$$

- Then

$a > 0$: Spiral source, unstable

$a = 0$: Periodic (an ellipse), stable

$a < 0$: Spiral sink, stable

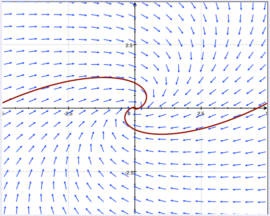
[2] See Euler's Formula ▶

Complex Possibilities

$\lambda_{\pm} = a \pm b\imath$

I. $a < 0$

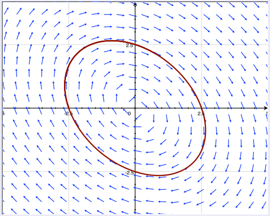
$$\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}$$



Spiral Sink, Stable

II. $a = 0$

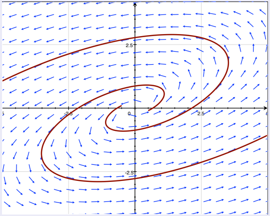
$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix}$$



Periodic

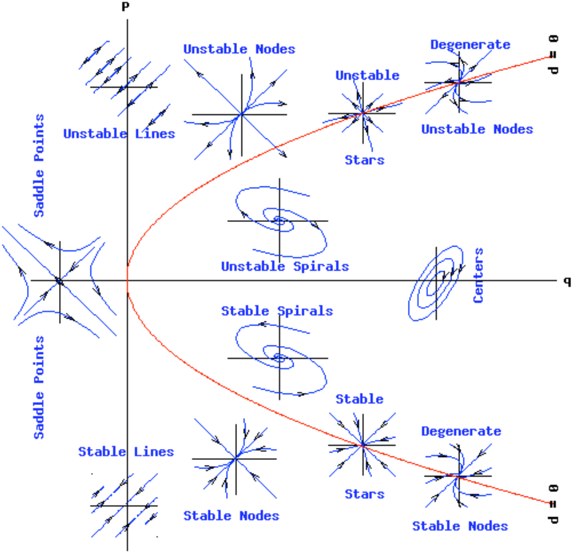
III. $a > 0$

$$\mathbf{A} = \begin{bmatrix} 3 & -6 \\ 2 & -1 \end{bmatrix}$$



Spiral Source, Unstable

Summary of the Summary



The parabola, *the discriminant of the characteristic equation*, is $p^2 - 4q = 0$ where $p = \mathrm{Tr}(\mathbf{A})$ and $q = \mathrm{Det}(\mathbf{A})$.

Sidebar: Euler's Formula

Euler's Formula

Leonhard Euler's formula[3] (c.1740)

$$e^{\imath x} = \cos(x) + \imath \sin(x),$$

for $x \in \mathbb{R}$, is a consequence of the identity $\imath x = \ln(\cos(x) + \imath \sin(x))$ that Roger Cotes discovered in 1714.

Proof.

1. Polar representation gives $z = r\cos(\theta) + r\imath \sin(\theta)$ in the complex plane $\mathbb{C}$
2. Differentiate w.r.t.θ so that $\frac{d}{d\theta} z = -r\sin(\theta) + r\imath\cos(\theta) = \imath z$
3. Separate variables: $dz/z = \imath\, d\theta$.
4. And integrate: $\ln(z) = \imath\theta + c$ giving $z = e^{\imath\theta + c} = c_1 e^{\imath\theta}$
5. The boundary condition of $z = r$ for $(r,\theta) = (r,0)$ shows $c_1 = r$. □

[3]Note: $e^{\imath x}$ is a periodic function!

◂ Back to Complex Eigenvalues